数理解析研究所講究録
595巻 1986年 143-192

Algebraic surfaces for regular systems of weights 「未完」

Dedicated to Professor Masayoshi NAGATA on the occasion of his sixtieth birthday

Kyoji Saito (斎藤恭司)

RIMS, Kyoto University

ABSTRACT: We costruct following families of surfaces by compactifying Milnor fibers.

i) 49 families of K3-surfaces with certain curve configulations, most of which admitt elliptic fibrations over $\mathbb{P}^1$.

ii) 9 families of algebraic surfaces of $\kappa = 1$, $q = 0$, $P_g = 1$ or 2 with elliptic fibrations over $\mathbb{P}^1$.

iii) 6 families of algebraic surfaces of general type satisfying the numerical equalitiy $P_g = [c_1^2/2] + 2$ for $c_1^2 = 1,1,2,2,3,5$.

(κ:=Kodaira dimension, P_g :=geometric genus, q:=irregularity, c_1^2:=second Chern number)

Contents

§1 Introduction

(1.1) Pinkham [20] gave an interpretation for the Arnold's strange duality [1], using comapactifications of 14 triangle singularities of Dolgachev [5], where the comactifications are K3 surfaces with certain curve configulations. Looijenga studied such compactifications in details for triangle and Fuchsian singularities [15,16], to describe possible singularities in the deformation of them.

Along similar idea, we study compactifications of some hypersurface singularities listed by regular systems of weights [24]. As a result we obtain 49 families of K3 surfaces with curve configulations for minimally elliptic singularities of Laufer, 9 families of elliptic surfaces of Kodaira dimension 1 and 6 families of surfaces of general type with the equality $P_g = [c_1^2/2]+2$. (See (1.6),(1.7),(1.8) and §'s 2,3,4)

One motivation of this paper is an attempt to extend examples of period maps associated to primitive forms(cf (3.6),[18],[26]), which were well understood only for simple and simple elliptic singularity cases.

(1.2) We briefly recall Pinkham's compactification $\tilde{\tilde{X}}_1$ at a special point 1 of the moduli S. A review on weighted homogeneous singularity of dim 2 and the construction of the family $\tilde{\tilde{X}}_t$ ($t\in S_{\geq 0}$) of the surfaces for the singularity are given in §5, which prepare notations and concepts for the paragraphs 2,3 and 4. Some readers may be suggested to go directly to §'s 2,3 and 4 and refer to §5 for notations.

(1.3) Let positive integers a,b,c and h with GCD(a,b,c,h)=1 ,called a reduced system of weights, be given. The hypersurface $X_0 := \{ (x,y,z)\in\mathbb{C}^3 : f(x,y,z)=0 \}$ for a weighted homogenous polynomial $f(x,y,z) = \sum_{ai+bj+ck=h} c_{ijk} x^i y^j z^k$ with coefficients generic in $\mathbb{C}$ has an isolated singular point at the origin 0, iff the following rational function in T dose not have a pole on the unit cicle $|T| = 1$ (cf [23]).

$$\chi(T) := T^{-h}\frac{(T^h-T^a)(T^h-T^b)(T^h-T^c)}{(T^a-1)(T^b-1)(T^c-1)}$$

We call such (a,b,c;h) a regular system of weights. Then $\chi(T)$ can be developed in,

$$\chi(T) = T^{m_1} + T^{m_2} \dots + T^{m_\mu}$$

for some integers $m_1,\dots,m_\mu$, called the exponents for (a,b,c;h). This establish a one to one correspondence between the hypersurface isolated singularity X_0 with a C*-action and the regualr system of weights up to a suitable equivalence. Here $\mu := \frac{(h-a)(h-b)(h-c)}{abc}$ is the Milnor number of the singularity. The smallest exponent = a+b+c-h =: ε is characterized by several means (for instance [8],[32],[23]), playing an important roll for X_0. For instance the singularity X_0 is a rational double point for $\varepsilon > 0$, a simplly elliptic singulariy for $\varepsilon = 0$, and a Fuchsian singularity for $\varepsilon = -1$.

(1.4) For a regular system of weights (a,b,c;h), let us consider the hypersurface

$$\bar{X}_1 := \{ (x:y:z:w)\in \mathbb{P}(a,b,c,1) : f(x,y,z) = w^h \},$$

where $\mathbb{P}(a,b,c,1) := (\mathbb{C}^4-(0))/((x,y,z,w)\sim(t^a x, t^b y, t^c z, tw)$ for $t\in\mathbb{C}^\times)$. $\bar{X}_1$ is a compactification of the Milnor fiber $X_1 := \{ (x,y,z)\in\mathbb{C}^3 : f(x,y,z) = 1 \}$ by adding a curve at infinity. Denote by $\tilde{\tilde{X}}_1$ the surface of the minimal resolution of the singularities of of $\bar{X}_1$ at infinity. Put $D_\infty := \tilde{\tilde{X}}_1 - X_1$ and call it the divisor at infinity, which defines a star froming dual graph with the central curve E .

For example, $\tilde{\tilde{X}}_1$ is a rational surface with $K^2 = 2$ for $\varepsilon > 0$, $\tilde{\tilde{X}}_1 = \bar{X}_1$ is a Del Pezzo surface for $\varepsilon = 0$, and $\tilde{\tilde{X}}_1$ is a K3 surface for $\varepsilon = -1$ (See for instance [ ][ ][ ][ ][ ].)

(1.5) After the above mentioned systems of weights (a,b,c;h) with $\varepsilon = 0$ or ± 1, we are interested in the following three extremal boundary cases in the present paper.

i) (a,b,c;h) having only one negative exponent ε without 0 exponent.

ii) (a,b,c;h) having only one negative exponent ε with some 0 exponents.

iii) (a,b,c;h) such that the smallest exponent $\varepsilon := a+b+c-h$ is equal to -2 .

(1.6) The surfaces $\tilde{\tilde{X}}_1$ for the first group (1.5) i) is studied in §2.

There are 49 = 22+7+8+2+7+3 such reduced regular systems of weights according as $\varepsilon = -1, -2, -3, -4, -5$ and -7 (See [24]). All these weights defines minimally elliptic singularities $\tilde{\tilde{X}}_0$ in the sence of Laufer [14] (cf. (5.7) iv) b)).

This group includes 22 systems of weights with $\varepsilon = -1$ for Fuchsian singularities, particularly 14 exceptional unimodular singularities. Including these Fuchsian cases, the surfaces $\tilde{\tilde{X}}_1$ for the group (1.5) i) have the following descriptions.

There is a maximal sub-configulation D_1 of D_∞ which can be blow down to a smooth point. The blow down surface $\tilde{\tilde{\tilde{X}}}_1 := \tilde{\tilde{X}}_1/D_1$ is a K3 surface with a curve configulation D_∞/D . (Particularly $D_1 = \phi$ for Fuchsian singularities.) There is a sub-configulation $\tilde{\tilde{\tilde{D}}}_2$ of D_∞/D_1 , whose linear system defines a fibration of $\tilde{\tilde{\tilde{X}}}_1$ over $\mathbb{P}^1$, most of which are elliptic fibrations.

The detailed descriptions of the divisor D_∞ and the fibration are given in § 2.

Note 1. Shioda's study on elliptic surfaces [29].

(1.7) The surfaces $\tilde{\tilde{X}}_1$ for the second group (1.5) ii) are studied in §3.

There are 12 = 9+2+1 reduced regular systems weights according as $\varepsilon = -1, -2$ or -3 for this group. The surface $\tilde{\tilde{X}}_1$ is already minimal whose Kodaira dimension κ is equal to 0 or 1 according as $\varepsilon = -1$ or less. The geometric genus p_g and the first Chern number c_2 of the surface are 1 and 0 respectively. The linear system $|-\varepsilon E_\infty|$ defines an elliptic fibration which admitts a global simple double or triple section according as $\varepsilon = -1, -2$ or -3. The details will be described in §3.

(1.8) The surfaces $\tilde{\tilde{X}}_t$ for $\varepsilon = -2$ of the group (1.5) iii) are studied in §4.

There are 21 reduced regular systems of weights with $\varepsilon = -2$. In this case the cannonical divisor of the surface is given by $K_{\tilde{\tilde{X}}_1} = E_\infty$, where E_∞ is smooth of genus a_0 and $E_\infty^2 = a_0 - 1$. Here a_0 is the multiplicity of 0 exponents.

Therefore the surfaces $\tilde{\tilde{X}}_t$ are classified according to a_0 as follows.

i) $a_0 = 0$: There are 7 regular systems of weights of this class. They belong to the class of (1.5) i) too, which are studied in §2. By blowing down the curve E_∞, one obtains a family of elliptic K3 surfaces as described there.

ii) $a_0 = 1$: There are 8 regular systems of weights of this class. Two of them belong to the class (1.5)ii) studied in §3. The remainings are surfces of Kodaira dimension $\kappa = 1$ with the irregularity $q = 0$ and $P_g = 1$. The linear system $|E_\infty|$ defines the elliptic fibration over $\mathbb{P}^1$ which has a global section.

iii) $a_0 > 1$: There are 6 regular systems of weights of this type. They give families of surfaces of general type. The pair (P_g, c_1^2) of the geometric genus and the second Chern number of $\tilde{\tilde{X}}_t$ are (4,5),(3,3),(3,2),(3,2),(2,1) and (2,1), which satisfy a relation $P_g = [c_1^2/2] + 2$. The linear system $|E_\infty|$ defines either a g=2 fibration, a triple or double covering or an embedding as a quintic surface.

The more detailed description of the surfaces is given in §4.

Note 1. These 21 regular systems of weights are naturally corresponding to co-compact subgroups Γ of $SL(2,\mathbb{R})$ satisfying $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \notin \Gamma$ (cf. (5.3) Note 2.).

Note 2. In general an inequality $P_g \leq [c_1^2/2] + 2$ holds. Those surfaces with the equality are studied by sevral authers Enriques, Noether, Moischezon, Horikawa, Todorov and others (cf [13],[32],[28]).

(1.9) The auther was supported by MPI for Math. in Bonn in Spring '85 when he was preparing this paper. He expresses his gratitude to Prof. Hirzebruch and the members of the institute for the hospiatlities and encouragings. He also expresses his gratitude to E. Brieskorn, E. Looijenga, I. Naruki, F. Sakai, E. Sato, A. Todorov, M. Tomari and J. Wahl for their inspiring discussions.

§ 2 The class having one negative exponent without 0 exponent

In this paragraph, we study the surfaces for regular systems of weights which has one negative exponent but no 0 exponent. The main results formulated in (2.5),(2.6) show that the most of them give families of elliptic K3 surfaces.

(2.1) Systems of weights for minimally elliptic singularities.

Consider a weighted homogeneous hypersurface isolated singular point at 0 in $\mathbb{C}^3$,

(2.1.1) $X_0 := (\ (x,y,z)\in\mathbb{C}^3 :\ f(x,y,z) = 0\ ),$

(2.1.2) $f(x,y,z) = \sum_{ai+bj+ck=h} c_{ijk}\, x^i y^j z^k$

Definition

where (a,b,c;h) is a reduced regular system of weights (cf. (1.3),(5.5)).

The singularity X_0 is minimally elliptic (characterized as $p_g = 1$, Laufer [14]), iff there exists one non-positive exponent for (a,b,c;h)(cf(5.7)iv)b)). The condition is equivalent that either one of the followings holds((5.5.7),[24 (4.3)]):

(2.1.3) i) $\varepsilon = -1$ and $\min(a,b,c) > -\varepsilon + 1$,

ii) $\min(a,b,c) = -\varepsilon + 1$.

The TABLE 1. is a recalling of the list of reduced regular systems of weights (a,b,c;h) satisfying i) or ii) from [24]. (The 14 systems of $\varepsilon = -1$ Type II in the table satisfy the innequality i) and all the remainings satisfy the equality ii).)

TABLE 1.

(a,b,c;h)	exponents
$\varepsilon = 0$	
(1,1,1;3)	0,1,1,1,2,2,2,3
(1,1,2;4)	0,1,1,2,2,2,3,3,4
(1,2,3;6)	0,1,2,2,3,3,4,4,5,6
$\varepsilon = -1$ Type I.	
(2,2,3;8)	-1,1,1,2,3,3,3,4,5,5,5,6,7,7,9
(2,2,5;10)	-1,1,1,3,3,3,5,5,5,5,8,8,8,9,9,11
(2,3,3;9)	-1,1,2,2,3,4,4,5,5,6,7,7,8,10
(2,3,4;10)	-1,1,2,3,3,4,5,5,6,7,7,8,9,11
(2,3,6;12)	-1,1,2,3,4,5,5,6,7,7,8,9,10,11,13
(2,4,5;12)	-1,1,3,3,4,5,5,7,7,8,9,9,11,13
(2,4,7;14)	-1,1,3,3,5,5,7,7,7,9,9,11,11,13,15
(2,6,9;18)	-1,1,3,5,5,7,7,9,9,11,11,13,13,15,17,19

ε = -1   Type II.

(3,4,4;12)	-1,2,3,3,5,6,6,7,9,9,10,13
(3,4,5;13)	-1,2,3,4,5,6,7,8,9,10,11,14
(4,5,6;16)	-1,3,4,5,7,8,9,11,12,13,17
(3,5,6;15)	-1,2,4,5,5,7,8,10,10,11,13,16
(4,6,7;18)	-1,3,5,6,7,9,11,12,13,15,19
(6,8,9;24)	-1,5,7,8,11,13,16,17,19,25
(3,4,8;16)	-1,2,3,5,6,7,8,9,10,11,13,14,17
(4,5,10;20)	-1,3,4,7,8,9,11,12,13,16,17,21
(3,5,9;18)	-1,2,4,5,7,8,9,10,11,13,14,16,19
(4,6,11;22)	-1,3,5,7,9,11,11,13,15,17,19,23
(6,8,15;30)	-1,5,7,11,13,15,17,19,23,25,31
(3,8,12;24)	-1,2,5,7,8,10,11,13,14,16,17,19,22,25
(4,10,15;30)	-1,3,7,9,11,13,15,17,19,21,23,27,31
(6,14,21;42)	-1,5,11,13,17,19,23,25,29,31,37,43

ε = -2

(3,3,4;12)	-2,1,1,2,4,4,4,5,5,7,7,8,8,8,10,11,11,14
(3,5,5;15)	-2,1,3,3,4,6,6,7,8,9,9,11,12,12,14,17
(3,5,7;17)	-2,1,3,4,5,6,7,8,9,10,11,12,13,14,16,19
(3,5,10;20)	-2,1,3,4,6,7,8,9,10,11,12,13,14,16,17,19,22
(3,7,9;21)	-2,1,4,5,7,7,8,10,11,13,14,14,16,17,20,23
(3,7,12;24)	-2,1,4,5,7,8,10,11,12,13,14,16,17,19,20,23,26
(3,10,15;30)	-2,1,4,7,8,10,11,13,14,16,17,19,20,22,23,26,29,32

ε = -3

(4,5,7;19)	-3,1,2,4,5,6,7,8,9,10,11,12,13,14,15,18,22
(4,5,8;20)	-3,1,2,5,5,6,7,9,10,10,11,13,14,15,15,18,19,23
(4,5,12;24)	-3,1,2,5,6,7,9,10,11,12,13,14,15,17,18,19,22,23,27
(4,7,10;24)	-3,1,4,5,7,8,9,11,12,13,15,16,17,19,20,23,27
(4,7,14;28)	-3,1,4,5,8,9,11,12,13,15,16,17,19,20,23,24,27,31
(4,10,13;30)	-3,1,5,7,9,10,11,13,15,17,19,20,21,23,25,29,33
(4,10,17;34)	-3,1,5,7,9,11,13,15,17,17,19,21,23,25,27,29,33,37
(4,14,21;42)	-3,1,5,9,11,13,15,17,19,21,23,25,27,29,31,33,37,41,45

$\varepsilon = -4$

(5,6,9;24)	-4,1,2,5,6,7,8,10,11,12,13,14,16,17,18,19,22,23,28
(5,6,15;30)	-4,1,2,6,7,8,11,12,13,14,16,17,18,19,22,23,24,28,29,34
$\varepsilon = -5$	
(6,7,9;27)	-5,1,2,4,7,8,9,10,11,13,14,16,17,18,19,20,23,25,26,32
(6,8,11;30)	-5,1,3,6,7,9,11,12,13,15,17,18,19,21,23,24,27,29,35
(6,8,13;32)	-5,1,3,7,8,9,11,13,15,16,17,19,21,23,24,25,29,31,37
(6,8,19;38)	-5,1,3,7,9,11,13,15,17,19,19,21,23,25,27,29,31,35,37,43
(6,16,21;48)	-5,1,7,11,13,16,17,19,23,25,29,31,32,35,37,41,47,53
(6,16,27;54)	-5,1,7,11,13,17,19,23,25,27,29,31,35,37,41,43,47,53,59
(6,22,33;66)	-5,1,7,13,17,19,23,25,29,31,35,37,41,43,47,49,53,59,65,71
$\varepsilon = -7$	
(8,9,12;36)	-7,1,2,5,9,10,11,13,14,17,18,19,22,23,25,26,27,31,34,35,43
(8,10,15;40)	-7,1,3,8,9,11,13,16,17,19,21,24,27,29,31,32,37,39,47
(8,10,25;50)	-7,1,3,9,11,13,17,19,21,23,25,27,29,31,33,37,39,41,47,49,57

(2.2) The polynomial $f(x,y,z,\lambda)$ and (m_+,m_0,m_-).

Let $f(x,y,z)$ be a weighted homogneous polynomial (2.1.2) having an isolated critical point at 0, for the system of weights (a,b,c;h) of TABLE 1.(cf (1.3). Laufer [14, appendix] has already listed such polynomial equations for minimally elliptic singularities. Among them, 3 cases for = 0 are simplly elliptic singularities [ ] and 14 cases for $\varepsilon = -1$ Type II. are exceptional unimodular singularities [ ]. In general, singularities for = -1 are called Fuchsian ([ ]).

In the TABLE 2. we recall and complete the list of polynomial $f(x,y,z,)$ with m -number of parameters $\lambda=(\lambda_1,\ldots,\lambda_{m_0})$, where m_+, m_0 and $m_-$are dimensions of positive, zero and negative graded part of the universal unfolding of f respectively (5.7.2).

The polynomials are normalized for a later application (see (2.4) Note.).

TABLE 2.

(a,b,c;h)	μ	m_-,m_0,m_+	polynomial	
$\varepsilon = 0$				
(1,1,1;3)	8	0,1,7	$x(x-y)(X-\lambda y) - yz$	$\lambda \neq 0,1.$
(1,1,2;4)	9	0,1,8	$xy(x-y)(x-\lambda y) - z$	$\lambda \neq 0,1.$
(1,2,3;6)	10	0,1,9	$y(x-y)(x-\lambda y) - z$	$\lambda \neq 0,1.$

$\varepsilon = -1$ Type I.

(2,2,3;8)	15	1,2,12	$x(x-y)(x-\lambda_1 y)(x-\lambda_2 y) + yz^2$	$\lambda_i \neq 0,1,\ \lambda_1 \neq \lambda_2$.
(2,2,5;10)	16	1,2,13	$xy(x-y)(x-\lambda_1 y)(x-\lambda_2 y) + z^2$	$\lambda_i \neq 0,1,\ \lambda_1 = \lambda_2$.
(2,3,3;9)	14	1,1,12	$x^3 y + z(z-y)(z-\lambda y)$	$\lambda \neq 0,1$.
(2,3,4;10)	14	1,1,12	$x(z-x^2)(z-\lambda x^2) - y^2 z$	$\lambda \neq 0,1$.
(2,3,6;12)	15	1,1,13	$(y^2-x^3)(y^2-\lambda x^3) + z^2$	$\lambda \neq 0,1$.
(2,4,5;12)	14	1,1,12	$y(y-x^2)(y-\lambda x^2) - xz^2$	$\lambda \neq 0,1$.
(2,4,7;14)	15	1,1,13	$xy(y-x^2)(y-\lambda x^2) - z^2$	$\lambda \neq 0,1$.
(2,6,9;18)	16	1,1,14	$y(y-x^3)(y-\lambda x^3) - z^2$	$\lambda \neq 0,1$.

$\varepsilon = -1$ Type II.

(3,4,4;12)	12	1,0,11	$x^4 + yz(y-z)$
(3,4,5;13)	12	1,0,11	$x^3 y + y^2 z + z^2 x$
(4,5,6;16)	11	1,0,10	$x^4 + y^2 z + z^2 x$
(3,5,6;15)	12	1,0,11	$x^3 z + y^3 + xz^2$
(4,6,7;18)	11	1,0,10	$x^3 y + y^3 + xz^2$
(6,8,9;24)	10	1,0,9	$x^4 + y^3 + xz^2$
(3,4,8;16)	13	1,0,12	$yx^4 + y^2 z + z^2$
(4,5,10;20)	12	1,0,11	$x^5 + y^2 z + z^2$
(3,5,9;18)	13	1,0,12	$x^3 z + xy^3 + z^2$
(4,6,11;22)	12	1,0,11	$yx^4 + xy^3 + z^2$
(6,8,15;30)	11	1,0,10	$x^5 + xy^3 + z^2$
(3,8,12;24)	14	1,0,13	$x^4 z + y^3 + z^2$
(4,10,15;30)	13	1,0,12	$yx^5 + y^3 + z^2$
(6,14,21;42)	12	1,0,11	$x^7 + y^3 + z^2$

$\varepsilon = -2$

(3,3,4;12)	18	3,1,14	$xy(x-y)(x-\lambda y) + z^3$	$\lambda \neq 0,1$.
(3,5,5;15)	16	2,0,14	$x^5 + yz(y-z)$	
(3,5,7;17)	16	2,0,14	$x^4 y + y^2 z + z^2 x$	
(3,5,10;20)	17	2,0,15	$x^5 y + y^2 z + z^2$	
(3,7,9;21)	16	2,0,14	$x^4 z + y^3 + z^2 x$	
(3,7,12;24)	17	2,0,15	$x^4 z + xy^3 + z^2$	
(3,10,15;30)	18	2,0,16	$x^5 z + y^3 + z^2$	

$\mathcal{E} = -3$			
(4,5,7;19)	18	3,0,15	$x^3z + y^3x + z^2y$
(4,5,8;20)	18	3,0,15	$x^3z + y^4 + z^2x$
(4,5,12;24)	19	3,0,16	$x^3z + y^4x + z^2$
(4,7,10;24)	17	2,0,15	$x^6 + y^2z + z^2x$
(4,7,14;28)	18	2,0,16	$x^7 + y^2z + z^2$
(4,10,13;30)	17	2,0,15	$x^5y + y^3 + z^2x$
(4,10,17;34)	18	2,0,16	$x^6y + y^3x + z^2$
(4,14,21;42)	19	2,0,17	$x^7y + y^3 + z^2$
$\mathcal{E} = -4$			
(5,6,9;24)	19	3,0,16	$x^3z + y^4 + z^2y$
(5,6,15;30)	20	3,0,17	$x^3z + y^5 + z^2$
$\mathcal{E} = -5$			
(6,7,9;27)	20	4,0,16	$x^3z + y^3x + z^3$
(6,8,11;30)	19	3,0,16	$x^5 + y^3x + z^2y$
(6,8,13;32)	19	3,0,16	$x^4y + y^4 + z^2x$
(6,8,19;38)	20	3,0,17	$x^5y + y^4x + z^2$
(6,16,21;48)	18	2,0,16	$x^8 + y^3 + z^2x$
(6,16,27;54)	19	2,0,17	$x^9 + y^3x + z^2$
(6,22,33;66)	20	2,0,18	$x^{11} + y^3 + z^2$
$\mathcal{E} = -7$			
(8,9,12;36)	21	4,0,17	$x^3z + y^4 + z^3$
(8,10,15;40)	20	3,0,17	$x^5 + y^4 + z^2y$
(8,10,25;50)	21	3,0,18	$x^5y + y^5 + z^2$

As a conseqence of the table we see and it is not hard to prove the following.

Assertion i) $m_- = \langle\, e \in (a,b,c) : e < -2\mathcal{E} \,\rangle + 1$,

$m_0 = \langle\, e \in (a,b,c) : e = -2\mathcal{E} \,\rangle$.

ii) The polynomial $f(x,y,z,\lambda)$ can be expressed as a sum of $m_0 + 3$ monomials in x,y and z. Particularly if $m_0 = 0$ (which is most of the cases), the polynomial $f(x,y,z)$ is unique up to automorphisms of the coordinate ring.

(Proof is a combination of (5.7.2), (2.1.3) and [23 (1.9.1),(3.6)].)

(2.3) From now on in this paper, we consider only the 49 cases with $\varepsilon < 0$. Note that the intersection form for the middle homology group of the Milnor fiber for this class of singularities has signature $(\mu_+, \mu_0, \mu_-) = (2, 0, \mu-2)$ (cf (5.7.4)).

(2.4) The minimal good resolution $\pi : \tilde{X}_0 \longrightarrow X_0$ of the singularity X_0 (2.1.1) is described in (5.6)(cf [6],[14],[19],[21]). The exceptional set $\pi^{-1}(0)$ defines a star shaped dual graph (5.6.1), whose central curve is denoted by E_0. The dual graph is numerically determined by the data: i) the genus of $E_0 = g(E_0)$, which is always 0 in this case ((5.6.3)) so that it will be omitted, ii) the self intersection number $= E_0^2 = -(1 + \#(\, e \in (a,b,c) : e = d+1\,))$ (cf (5.6.4)), iii) the set $A := (p_1, \ldots, p_r)$ of the orders of the cyclic isotropy subgroups of the Fuchsian group Γ at the branching points on E_0 (5.6.5), iv) the number $d := h-a-b-c = -\varepsilon$. (The p_i's for the 14 exceptional singularities are well known as Dolgachev numbers.)

Furthermore the analytic data of the resolution is determined by the positions of the branching points on $E_0 = \mathbb{P}^1$. Hence we give a rational parametrization:

$$\begin{array}{ccl} \mathbb{P}^1 & \simeq & E_0 = (\ (x:y:z) \in \mathbb{P}(a,b,c) \ :\ f(x,y,z,\lambda) = 0\ ) \\ t & \longmapsto & (x:y:z) \end{array}$$

of the central curve E_0.

We shall describe in the TABLE 3. the following data for every regular systems of weights (a,b,c;h) of the TABLE 1..

i) The set $A := (\, p_1, \ldots, p_r)$.

ii) Polynomial presentation $(x(t), y(t), z(t))$ of the parametrization: $\mathbb{P}^1 \to E_0$.

iii) The values t_i of t at the branching points p_i on E_0.

iv) The order of zeros (n_{xi}, n_{yi}, n_{zi}) of $(x(t), y(t), z(t))$ at the branching point p_i.

v) The dual graph of the exceptional set $\pi^{-1}(0)$.

(We used p_i's$\in A$ as for the identification of the branching points on E_0.)

Note. In the TABLE 3. the polynomials $f(x,y,z,\lambda)$, $x(t)$, $y(t)$ and $z(t)$ are normalized as follows. (Recall that the branching points lie on the coordinate axis (5.6).)

i) (the values of t at branching points of E_0) = (the roots of $x(t)y(t)z(t)=0) \cup (\infty)$.

ii) $0 < n_{xi} \le a$, $0 < n_{yi} \le b$ and $0 < n_{zi} \le c$ for $p_i \in A = (p, \ldots, p)$.

iii) $n_{x\infty}$, $n_{y\infty}$ and $n_{z\infty}$ are defined by the following relation.

$$\sum_{i=1}^{r} \begin{bmatrix} n_{xi} \\ n_{yi} \\ n_{zi} \end{bmatrix} = (m_0 + 1)\begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{for } \varepsilon = -1, \text{ or } = (m + 2)\begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{for } \varepsilon \lesssim -2.$$

TABLE 3.

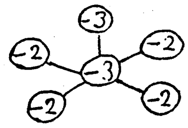
(a,b,c;h) $\quad A:=(p_1,\ldots,p_r)\quad$ parametrization of E_0 $\quad$ dual graph

$\mathcal{E} = -1$ Type I.

(2,2,3;8) $\quad$ 2,2,2,2,3

$$\begin{array}{lccccc} t = & 0, & 1, & \lambda_1, & \lambda_2, & \infty \\ n_x = & 2 & 1 & 1 & 1 & 1 \\ n_y = & 1 & 1 & 1 & 1 & 2 \\ n_z = & 2 & 2 & 2 & 2 & 1 \end{array}$$

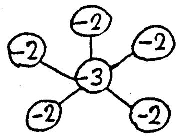
$$x = -t^2(t-1)(t-\lambda_1)(t-\lambda_2),\quad y = -t(t-1)(t-\lambda_1)(t-\lambda_2),\quad z = \pm t^2(t-1)^2(t-\lambda_1)^2(t-\lambda_2)^2.$$



(2,2,5;10) $\quad$ 2,2,2,2,2

$$\begin{array}{lccccc} t = & 0, & 1, & \lambda_1, & \lambda_2, & \infty \\ n_x = & 2 & 1 & 1 & 1 & 1 \\ n_y = & 1 & 1 & 1 & 1 & 2 \\ n_z = & 3 & 3 & 3 & 3 & 3 \end{array}$$

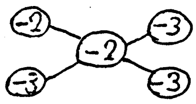
$$x = -t^2(t-1)(t-\lambda_1)(t-\lambda_2),\quad y = -t(t-1)(t-\lambda_1)(t-\lambda_2),\quad z = \pm t^3(t-1)^3(t-\lambda_1)^3(t-\lambda_2)^3.$$



(2,3,3;9) $\quad$ 2,3,3,3

$$\begin{array}{lcccc} t = & \infty, & 0, & 1, & \lambda \\ n_x = & 1 & 1 & 1 & 1 \\ n_y = & 3 & 1 & 1 & 1 \\ n_z = & 3 & 2 & 1 & 1 \end{array}$$

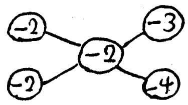
$$x = -t(t-1)(t-\lambda),\quad y = \pm t(t-1)(t-\lambda),\quad z = \pm t^2(t-1)(t-\lambda).$$



(2,3,4;10) $\quad$ 2,2,3,4

$$\begin{array}{lcccc} t = & 1, & \lambda, & \infty, & 0 \\ n_x = & 1 & 1 & 1 & 1 \\ n_y = & 2 & 2 & 1 & 1 \\ n_z = & 2 & 2 & 1 & 3 \end{array}$$

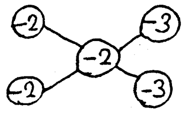
$$x = -t(t-1)(t-\lambda),\quad y = \pm t(t-1)^2(t-\lambda)^2,\quad z = t^3(t-1)^2(t-\lambda)^2.$$



(2,3,6;12) $\quad$ 2,2,3,3

$$\begin{array}{lcccc} t = & \infty, & 0, & 1, & \lambda \\ n_x = & 1 & 1 & 1 & 1 \\ n_y = & 2 & 2 & 1 & 1 \\ n_z = & 4 & 3 & 3 & 2 \end{array}$$

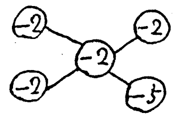
$$x = t(t-1)(t-\lambda),\quad y = \pm t^2(t-1)(t-\lambda),\quad z = \lambda t^3(t-1)^3(t-\lambda)^2.$$



(2,4,5;12) $\quad$ 2,2,2,5

$$\begin{array}{lcccc} t = & 0, & 1, & \lambda, & \infty \\ n_x = & 1 & 1 & 1 & 1 \\ n_y = & 3 & 2 & 2 & 1 \\ n_z = & 3 & 3 & 3 & 1 \end{array}$$

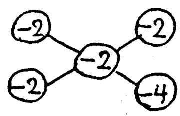
$$x = -t(t-1)(t-\lambda),\quad y = t^3(t-1)^2(t-\lambda)^2,\quad z = \pm t^3(t-1)^3(t-\lambda)^3.$$



(2,4,7;14) $\quad$ 2,2,2,4

$$\begin{array}{lcccc} t = & 0, & 1, & \lambda, & \infty \\ n_x = & 1 & 1 & 1 & 1 \\ n_y = & 3 & 2 & 2 & 1 \\ n_z = & 4 & 4 & 4 & 2 \end{array}$$

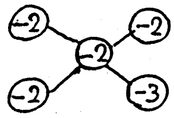
$$x = -t(t-1)(t-\lambda),\quad y = t^3(t-1)^2(t-\lambda)^2,\quad z = \pm t^4(t-1)^4(t-\lambda)^4.$$



(2,6,9;18) $\quad$ 2,2,2,3

$$\begin{array}{lcccc} t = & 0, & 1, & \lambda, & \infty \\ n_x = & 1 & 1 & 1 & 1 \\ n_y = & 4 & 3 & 3 & 2 \\ n_z = & 5 & 5 & 5 & 3 \end{array}$$

$$x = -t(t-1)(t-\lambda),\quad y = -t^4(t-1)^3(t-\lambda)^3,\quad z = \pm t^5(t-1)^5(t-\lambda)^5.$$

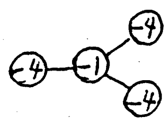


Type II.

(3,4,4;12) $\quad$ 4, 4, 4

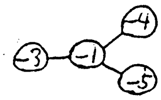
$$\begin{array}{lccc} t = & \infty, & 1, & 0 \\ n_x = & 1 & 1 & 1 \\ n_y = & 1 & 1 & 2 \\ n_z = & 1 & 2 & 1 \end{array}$$

$$x = \pm t(t-1),\quad y = -t^2(t-1),\quad z = -t(t-1)^2.$$



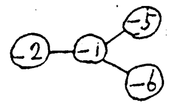
(3,4,5;13)

	3,	4,	5	
$t =$	∞,	1,	0	
$n_x =$	1	1	1	$x = \pm t\ (t-1)$,
$n_y =$	2	1	1	$y = -t\ (t-1)$,
$n_z =$	2	2	1	$z = \pm t\ (t-1)^2$.



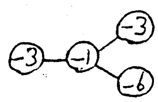
(4,5,6;16)

	2,	5,	6	
$t =$	∞,	1,	0	
$n_x =$	2	1	1	$x = -t\ (t-1)$,
$n_y =$	3	1	1	$y = \pm t\ (t-1)$,
$n_z =$	3	2	1	$z = -t\ (t-1)^2$.



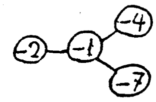
(3,5,6;15)

	3,	3,	6	
$t =$	∞,	1,	0	
$n_x =$	1	1	1	$x = \pm t\ (t-1)$,
$n_y =$	2	2	1	$y = \pm t\ (t-1)^2$,
$n_z =$	3	2	1	$z = -t\ (t-1)^2$.



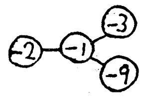
(4,6,7;18)

	2,	4,	7	
$t =$	∞,	1,	0	
$n_x =$	2	1	1	$x = -t\ (t-1)$,
$n_y =$	3	2	1	$y = -t\ (t-1)^2$,
$n_z =$	4	2	1	$z = \pm t\ (t-1)^2$.



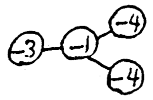
(6,8,9;24)

	2,	3,	9	
$t =$	∞,	1,	0	
$n_x =$	3	2	1	$x = -t\ (t-1)^2$,
$n_y =$	4	3	1	$y = -t\ (t-1)^3$,
$n_z =$	5	3	1	$z = \pm t\ (t-1)^3$.



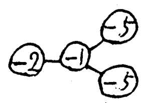
(3,4,8;16)

	3,	4,	4	
$t =$	∞,	1,	0	
$n_x =$	1	1	1	$x = \pm t\ (t-1)$,
$n_y =$	2	1	1	$y = -t\ (t-1)$,
$n_z =$	3	2	3	$z = -t^3 (t-1)^2$.



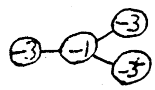
(4,5,10;20)

	2,	5,	5	
$t =$	∞,	1,	0	
$n_x =$	2	1	1	$x = -t\ (t-1)$,
$n_y =$	3	1	1	$y = \pm t\ (t-1)$,
$n_z =$	5	2	3	$z = -t^3 (t-1)^2$.



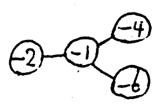
(3,5,9;18)

	3,	3,	5	
$t =$	∞,	1,	0	
$n_x =$	1	1	1	$x = \pm t\ (t-1)$,
$n_y =$	2	2	1	$y = \pm t\ (t-1)^2$,
$n_z =$	4	3	2	$z = \mp t^2 (t-1)^3$.



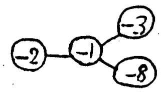
(4,6,11;22)

	2,	4,	6	
$t =$	∞,	1,	0	
$n_x =$	2	1	1	$x = -t\ (t-1)$,
$n_y =$	3	2	1	$y = -t\ (t-1)^2$,
$n_z =$	6	3	2	$z = \pm t^2 (t-1)^3$.



(6,8,15;30)

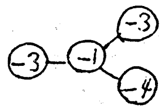
	2,	3,	8	
$t =$	∞,	1,	0	
$n_x =$	3	2	1	$x = -t\ (t-1)^2$,
$n_y =$	4	3	1	$y = -t\ (t-1)^3$,
$n_z =$	8	5	2	$z = \pm t^2 (t-1)^5$.



(3,8,12;24)

$$\begin{array}{llll} & 3, & 3, & 4 \\ t = & \infty, & 1, & 0 \\ n_x = & 1 & 1 & 1 \\ n_y = & 3 & 3 & 2 \\ n_z = & 5 & 4 & 3 \end{array}$$

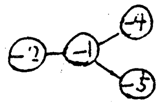
$x = \pm t\,(t-1)$, $y = t^2(t-1)^3$, $z = -t^3(t-1)^4$.



(4,10,15;30)

$$\begin{array}{llll} & 2, & 4, & 5 \\ t = & \infty, & 1, & 0 \\ n_x = & 2 & 1 & 1 \\ n_y = & 5 & 3 & 2 \\ n_z = & 6 & 4 & 3 \end{array}$$

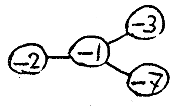
$z = -t\,(t-1)$, $y = t^2(t-1)^3$, $z = \pm t^3(t-1)^4$.



(6,14,21;42)

$$\begin{array}{llll} & 2, & 3, & 7 \\ t = & \infty, & 1, & 0 \\ n_x = & 3 & 2 & 1 \\ n_y = & 7 & 5 & 2 \\ n_z = & 11 & 7 & 3 \end{array}$$

$x = -t\,(t-1)^2$, $y = t^2(t-1)^5$, $z = \pm t^3(t-1)^7$.

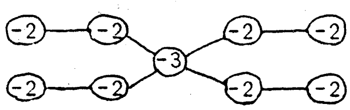


$\varepsilon = -2$

(3,3,4;12)

$$\begin{array}{lllll} & 3, & 3, & 3, & 3 \\ t = & 0, & 1, & \lambda, & \infty \\ n_x = & 3 & 2 & 2 & 2 \\ n_y = & 2 & 2 & 2 & 3 \\ n_z = & 3 & 3 & 3 & 3 \end{array}$$

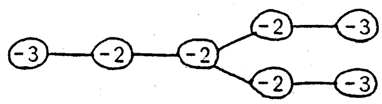
$x = \pm t^3(t-1)^2(t-\lambda)^2$, $y = \pm t^2(t-1)^2(t-\lambda)^2$, $z = -t^3(t-1)^3(t-\lambda)^3$.



(3,5,5;15)

$$\begin{array}{llll} & 5, & 5, & 5 \\ t = & \infty, & 1, & 0 \\ n_x = & 2 & 2 & 2 \\ n_y = & 3 & 3 & 4 \\ n_z = & 3 & 4 & 3 \end{array}$$

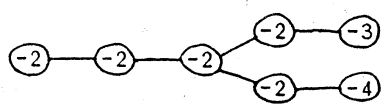
$x = \pm t^2(t-1)^2$, $y = \mp t^4(t-1)^3$, $z = \mp t^3(t-1)^4$.



(3,5,7;17)

$$\begin{array}{llll} & 7, & 5, & 3 \\ t = & 0, & 1, & \infty \\ n_x = & 2 & 2 & 2 \\ n_y = & 3 & 3 & 4 \\ n_z = & 4 & 5 & 5 \end{array}$$

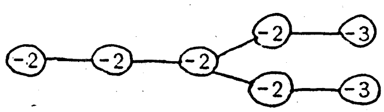
$x = \pm t^2(t-1)^2$, $y = \mp t^3(t-1)^3$, $z = \pm t^4(t-1)^5$.



(3,5,10;20)

$$\begin{array}{llll} & 5, & 5, & 3 \\ t = & 0, & 1, & \infty \\ n_x = & 2 & 2 & 2 \\ n_y = & 3 & 3 & 4 \\ n_z = & 7 & 6 & 7 \end{array}$$

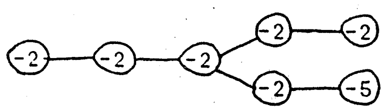
$x = \pm t^2(t-1)^2$, $y = t^3(t-1)^3$, $z = t^7(t-1)^6$.



(3,7,9;21)

$$\begin{array}{llll} & 9, & 3, & 3 \\ t = & 0, & 1, & \infty \\ n_x = & 2 & 2 & 2 \\ n_y = & 4 & 5 & 5 \\ n_z = & 5 & 6 & 7 \end{array}$$

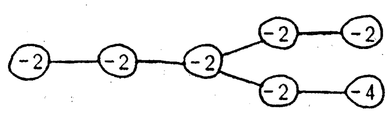
$x = \pm t^2(t-1)^2$, $y = \pm t^4(t-1)^5$, $z = \mp t^5(t-1)^6$.



(3,7,12;24)

$$\begin{array}{llll} & 7, & 3, & 3 \\ t = & 0, & 1, & \infty \\ n_x = & 2 & 2 & 2 \\ n_y = & 4 & 5 & 5 \\ n_z = & 7 & 8 & 9 \end{array}$$

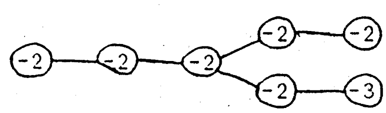
$x = \pm t^2(t-1)^2$, $y = \pm t^4(t-1)^5$, $z = -t^7(t-1)^8$.



(3,10,15;30)

$$\begin{array}{llll} & 5, & 3, & 3 \\ t = & 0, & 1, & \infty \\ n_x = & 2 & 2 & 2 \\ n_y = & 6 & 7 & 7 \\ n_z = & 9 & 10 & 11 \end{array}$$

$x = \pm t^2(t-1)^2$, $y = t^6(t-1)^7$, $z = \mp t^9(t-1)^{10}$.



$\varepsilon = -3$

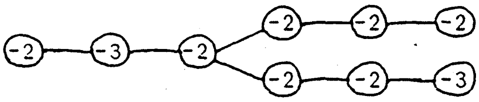
(4,5,7;19) 7, 5, 4

$t = 0,\ 1,\ \infty$

$n_x = 3\ \ 2\ \ 3 \qquad x = t^3(t-1)^2,$

$n_y = 4\ \ 2\ \ 4 \qquad y = \pm t^4(t-1)^2,$

$n_z = 5\ \ 3\ \ 6 \qquad z = \mp t^5(t-1)^3.$



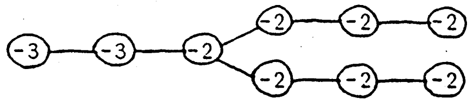
(4,5,8;20) 8, 4, 4

$t = 0,\ 1,\ \infty$

$n_x = 2\ \ 3\ \ 3 \qquad x = t^2(t-1)^3,$

$n_y = 2\ \ 4\ \ 4 \qquad y = \pm t^2(t-1)^4,$

$n_z = 3\ \ 6\ \ 7 \qquad z = -t^3(t-1)^6.$



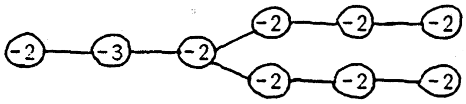
(4,5,12;24) 5, 4, 4

$t = 0,\ 1,\ \infty$

$n_x = 2\ \ 3\ \ 3 \qquad x = t^2(t-1)^3,$

$n_y = 2\ \ 4\ \ 4 \qquad y = \pm t^2(t-1)^4,$

$n_z = 5\ \ 9\ \ 10 \qquad z = -t^5(t-1)^9.$



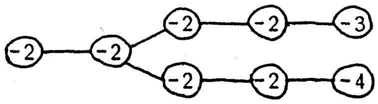
(4,7,10;24) 10, 7, 2

$t = 0,\ 1,\ \infty$

$n_x = 3\ \ 3\ \ 2 \qquad x = -t^3(t-1)^3,$

$n_y = 5\ \ 5\ \ 4 \qquad y = \pm t^5(t-1)^5,$

$n_z = 7\ \ 8\ \ 5 \qquad z = -t^7(t-1)^8.$



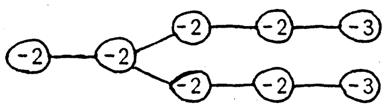
(4,7,14;28) 7, 7, 2

$t = 0,\ 1,\ \infty$

$n_x = 3\ \ 3\ \ 2 \qquad x = -t^3(t-1)^3,$

$n_y = 5\ \ 5\ \ 4 \qquad y = \pm t^5(t-1)^5,$

$n_z = 10\ \ 11\ \ 7 \qquad z = t^{10}(t-1)^{11}.$



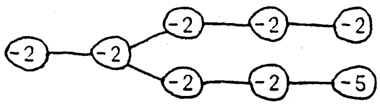
(4,10,13;30) 13, 4, 2

$t = 0,\ 1,\ \infty$

$n_x = 3\ \ 3\ \ 2 \qquad x = -t^3(t-1)^3,$

$n_y = 7\ \ 8\ \ 5 \qquad y = -t^7(t-1)^8,$

$n_z = 9\ \ 10\ \ 7 \qquad z = \pm t^9(t-1)^{10}.$



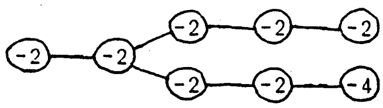
(4,10,17;34) 10, 4, 2

$t = 0,\ 1,\ \infty$

$n_x = 3\ \ 3\ \ 2 \qquad x = -t^3(t-1)^3,$

$n_y = 7\ \ 8\ \ 5 \qquad y = -t^7(t-1)^8,$

$n_z = 12\ \ 13\ \ 9 \qquad z = \pm t^{12}(t-1)^{13}.$



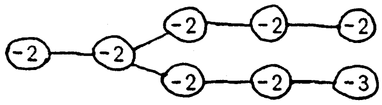
(4,14,21;42) 7, 4, 2

$t = \infty,\ 1,\ 0$

$n_x = 3\ \ 3\ \ 2 \qquad x = t^2(t-1)^3,$

$n_y = 10\ \ 11\ \ 7 \qquad y = t^7(t-1)^{11},$

$n_z = 15\ \ 16\ \ 11 \qquad z = \pm t^{11}(t-1)^{16}.$



$\varepsilon = -4$

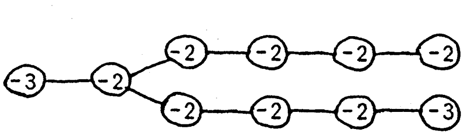
(5,6,9;24) 9, 5, 3

$t = \infty,\ 1,\ 0$

$n_x = 4\ \ 4\ \ 2 \qquad x = \pm t^2(t-1)^4,$

$n_y = 5\ \ 5\ \ 2 \qquad y = t^2(t-1)^5,$

$n_z = 7\ \ 8\ \ 3 \qquad z = \mp t^3(t-1)^8.$



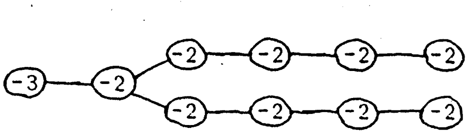
(5,6,15;30) 5, 5, 3

$t = 0,\ 1,\ \infty$

$n_x = 4\ \ 4\ \ 2 \qquad x = \pm t^4(t-1)^4,$

$n_y = 5\ \ 5\ \ 2 \qquad y = t^5(t-1)^5,$

$n_z = 13\ \ 12\ \ 5 \qquad z = \mp t^{13}(t-1)^{12}.$



$\varepsilon = -5$

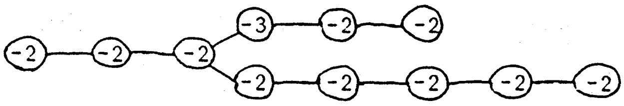
(6,7,9;27) 7, 6, 3

$t = 0,\ 1,\ \infty$

$n_x = 3\quad 5\quad 4 \qquad x = -t^3(t-1)^5,$

$n_y = 3\quad 6\quad 5 \qquad y = \pm t^3(t-1)^6,$

$n_z = 4\quad 8\quad 6 \qquad z = \mp t^4(t-1)^8.$



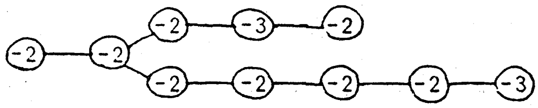
(6,8,11;30) 11, 8, 2

$t = \infty,\ 1,\ 0$

$n_x = 5\quad 4\quad 3 \qquad x = -t^3(t-1)^4,$

$n_y = 7\quad 5\quad 4 \qquad y = t^4(t-1)^5,$

$n_z = 9\quad 7\quad 6 \qquad z = \pm t^6(t-1)^7.$



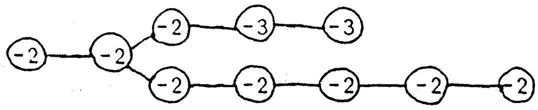
(6,8,13;32) 13, 6, 2

$t = 0,\ 1,\ \infty$

$n_x = 4\quad 5\quad 3 \qquad x = t^4(t-1)^5,$

$n_y = 5\quad 7\quad 4 \qquad y = -t^5(t-1)^7,$

$n_z = 8\quad 11\quad 7 \qquad z = \pm t^8(t-1)^{11}.$



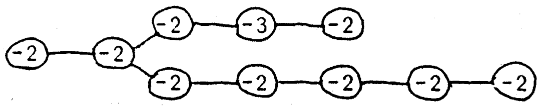
(6,8,19;38) 8, 6, 2

$t = 0,\ 1,\ \infty$

$n_x = 4\quad 5\quad 3 \qquad x = t^4(t-1)^5,$

$n_y = 5\quad 7\quad 4 \qquad y = -t^5(t-1)^7,$

$n_z = 12\quad 16\quad 10 \qquad z = \pm t^{12}(t-1)^{16}.$



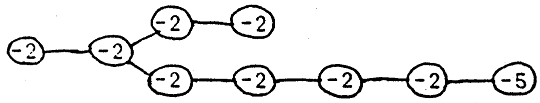
(6,16,21;48) 21, 3, 2

$t = 0,\ 1,\ \infty$

$n_x = 5\quad 4\quad 3 \qquad x = -t^5(t-1)^4,$

$n_y = 13\quad 11\quad 8 \qquad y = -t^{13}(t-1)^{11},$

$n_z = 17\quad 14\quad 11 \qquad z = \pm t^{17}(t-1)^{14}.$



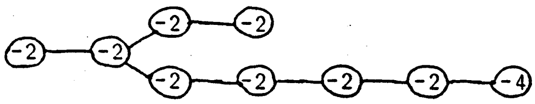
(6,16,27;54) 16, 3, 2

$t = 0,\ 1,\ \infty$

$n_x = 5\quad 4\quad 3 \qquad x = -t^5(t-1)^4,$

$n_y = 13\quad 11\quad 8 \qquad y = -t^{13}(t-1)^{11},$

$n_z = 22\quad 18\quad 14 \qquad z = \pm t^{22}(t-1)^{18}.$



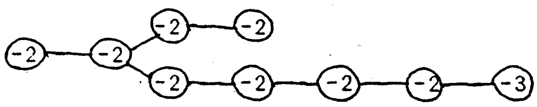
(6,22,33;66) 11, 3, 2

$t = 0,\ 1,\ \infty$

$n_x = 5\quad 4\quad 3 \qquad x = -t^5(t-1)^4,$

$n_y = 18\quad 15\quad 11 \qquad y = t^{18}(t-1)^{15},$

$n_z = 27\quad 22\quad 17 \qquad z = \pm t^{27}(t-1)^{22}.$



$\varepsilon = -7$

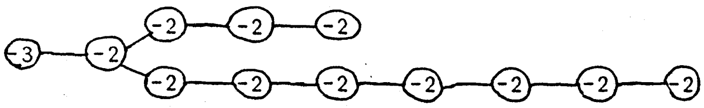
(8,9,12;36) 8, 4, 3

$t = 0,\ 1,\ \infty$

$n_x = 7\quad 6\quad 3 \qquad x = -t^7(t-1)^6,$

$n_y = 8\quad 7\quad 3 \qquad y = \pm t^8(t-1)^7,$

$n_z = 11\quad 9\quad 4 \qquad z = -t^{11}(t-1)^9.$



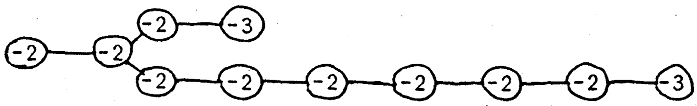
(8,10,15;40) 15, 5, 2

$t = 0,\ 1,\ \infty$

$n_x = 7\quad 5\quad 4 \qquad x = -t^7(t-1)^5,$

$n_y = 9\quad 6\quad 5 \qquad y = -t^9(t-1)^6,$

$n_z = 13\quad 9\quad 8 \qquad z = \pm t^{13}(t-1)^9.$



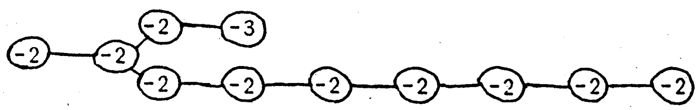
(8,10,25;50) 8, 5, 2

$t = 0,\ 1,\ \infty$

$n_x = 7\quad 5\quad 4 \qquad x = -t^7(t-1)^5,$

$n_y = 9\quad 6\quad 5 \qquad y = -t^9(t-1)^6,$

$n_z = 22\quad 15\quad 13 \qquad z = \pm t^{22}(t-1)^{15}.$



As a consequence of the above calculations, we obtain the following.

Assertion i) The number r of the branches of the resolution graph is given by

$$r = m_0 + 3 .$$

ii) The coordinates of the branching points on the central curve E_0 can be chosen to be $0, 1, \infty, \lambda_1, \ldots, \lambda_{m_0}$ where $(\lambda_1, \ldots, \lambda_{m_0})$ is the coordinates for the S_0(= the degree 0 part of the universal unfolding of f (cf. (5.7) i)) used in the TABLE 2. .

iii)

$$\det.\begin{pmatrix} a & n_{x0} & n_{x1} \\ b & n_{y0} & n_{y1} \\ c & n_{z0} & n_{z1} \end{pmatrix} = \pm 1$$

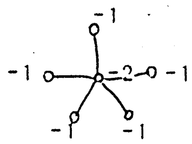
iv) The shape of the resolution graph, forgetting about the self-intersections of the components, depends only on the integers m_- , m_0 and $\varepsilon := a+b+c-h$ (= $-d$).

v) The cannonical diviser $K_{\tilde{X}_0}$ (5.6.7) depends only on the shape of the graph.

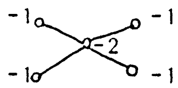
In the following TABLE 4. we list the shape of the dual graph and the coefficients of the cannonical diviser for the minimal good resolution above.

TABLE 4.

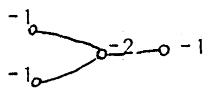
$\varepsilon = -1,\ m_- = 1,\ m_0 = 2$



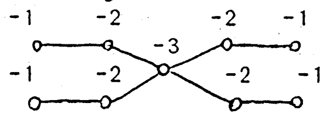
$\varepsilon = -1,\ m_- = 1,\ m_0 = 1$



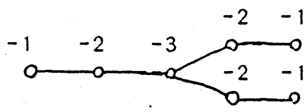
$\varepsilon = -1,\ m_- = 1,\ m_0 = 0$



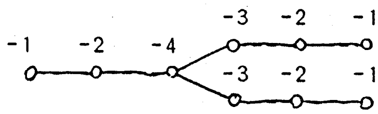
$\varepsilon = -2,\ m_- = 3,\ m_0 = 1$



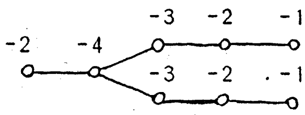
$\varepsilon = -2,\ m_- = 2,\ m_0 = 0$



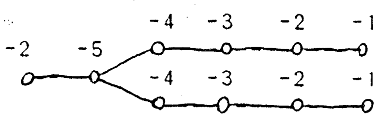
$\varepsilon = -3,\ m_- = 3,\ m_0 = 0$



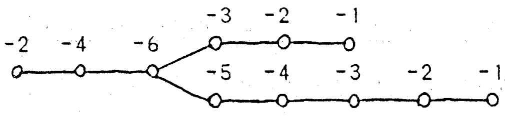
$\varepsilon = -3,\ m_- = 2,\ m_0 = 0$



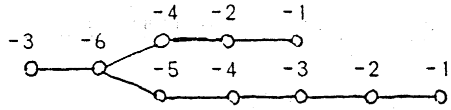
$\varepsilon = -4,\ m_- = 3,\ m_0 = 0$



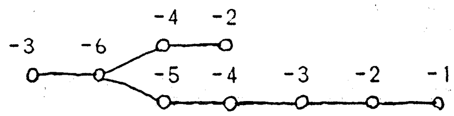
$\varepsilon = -5,\ m_- = 4,\ m_0 = 0$



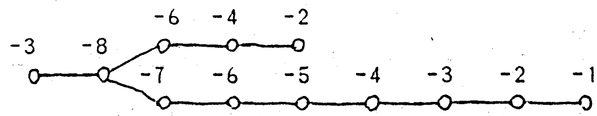
$\varepsilon = -5,\ m_- = 3,\ m_0 = 0$



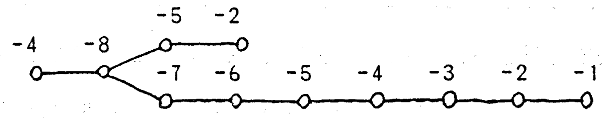
$\varepsilon = -5,\ m_- = 2,\ m_0 = 0$



$\varepsilon = -7,\ m_- = 4,\ m_0 = 0$



$\varepsilon = -7,\ m_- = 3,\ m_0 = 0$



Note. Many of the above graphs have the figure of affine Coxeter graphs of types $\tilde{D}_4$, $\tilde{E}_k$ (k=6,7,8). Such singlarities (called Kodaira sing.) are studied in [10].

(2.5) Compactifications.

The compactification $\tilde{\tilde{X}}_t$ of a Milnor fiber X_t for $t \in S$ is described in (5.8). Recall that $\tilde{\tilde{X}}_t = \tilde{X}_t \cup D_\infty$ (5.8.3), where $\tilde{X}_t$ is the minimal resolution of the affine variety X_t (5.7.3) and D_∞ is the divisor at infinity (5.8.4). The cannonical divisor of $\tilde{\tilde{X}}_t$ is $K_{\tilde{\tilde{X}}_t} = K_\infty + \sum_{x \in X_t} K_x$ where K_∞ is the cannonical divisor at infinity and K_x is the cannonical diviser of the resolution $\tilde{X}_t \longrightarrow X_t$ of a singular point x on X_t (5.8.7).

In this paragraph in TABLEs 5,6. we shall describe D_∞ and K_∞ expliciteIy. Before giving the TABLEs, we summerize some of their structures in the following Theorem, which implies that a minimal model $\tilde{\tilde{\tilde{X}}}_t$ of $\tilde{\tilde{X}}_t$ is a K3 surface for $t \in S_+$.

Theorem Let (a,b,c;h) be a regular system of weights of TABLE 1. Let $(\tilde{\tilde{X}}_t, D_\infty)$ for $t \in S_+$ be a pair of the compact smooth surface and its divisor at infinity for (a,b,c;h) as described in (5.8). Then the divisor D_∞ has the follwing decomposition.

(2.5.1) $$D_\infty = D_1 \cup D_2 \cup D_3$$

with the following properties:

i) The divisor D_1 in $\tilde{\tilde{X}}_t$ can be blow down to a smooth point. Let us denote by $\pi : \tilde{\tilde{X}}_t \longrightarrow \tilde{\tilde{\tilde{X}}}_t$ the blow down map, where $\tilde{\tilde{\tilde{X}}}_t := \tilde{\tilde{X}}_t / D_1$ is the smooth surface.

ii) The cannonical divisor K_∞ is equal to the cannonical divisor of the map π. (I.e. $K_\infty = \mathrm{div}(\pi^*(\omega))$ for a nonvanishing holomorphic 2-form ω on $\tilde{\tilde{X}}_t$ near the point $\pi(D_1)$.) This is equivalent to say that the cannonical divisor $K_{\tilde{\tilde{X}}}$ of $\tilde{\tilde{X}}_t$ is given by

$$(2.5.2) \qquad K_{\tilde{\tilde{X}}} = \sum_{x \in X_t} K_x \quad ,$$

where the sumation is over singularities of the affine surface X_t (cf.(5.8.7)).

iii) Put $\tilde{\tilde{D}}_2 := \pi(D_2)$. Then $\tilde{\tilde{D}}_2$ is either one of the followings.

a) A system of smooth rational curves whose intersection diagram is $\tilde{D}_4$ or $\tilde{E}_k$ (k=6,7,

b) Three smooth rational curves intersecting at a point normally each other.

c) Two smooth rational curves contacting at a point of order 2 or 3.

d) One rational curve with a cusp singular point of type (2,3),(2,5) or (3,4).

(Here (p,q)-cusp is a plane curve singularity, locally given by a equation $x^p - y^q = 0$.)

iv) The complete linear system $|\tilde{\tilde{D}}_2|$ in $\tilde{\tilde{X}}_t$ defines a fibration of $\tilde{\tilde{X}}_t$ over $\mathbb{P}^1$, most of which are elliptic fibrations. (For exact descriptions, see (2.6).)

v) $\tilde{\tilde{D}}_3 := \pi(D_3)$ is a union of smooth rational curves of selfintersections -2, whose connected components are of types either A_1, A_2, or A_3.

Corollary The surface $\tilde{\tilde{X}}_t$ is a K3 surface with a curve configulation $D_\infty / D_1 = \tilde{\tilde{D}}_2 \cup \tilde{\tilde{D}}_3$ for $t \in S_f$ (the rational double point part(cf (5.7)ii)). Hence the middle homology group $H_2(X_t, \mathbb{Z})$ of a Milnor fiber of the polynomial of TABLE 2. is embedded in the lattice of the K3 surface as an orthogonal complement of the classes of $\tilde{\tilde{D}}_2 \cup \tilde{\tilde{D}}_3$.

$$(2.5.3) \qquad H_2(X_t, \mathbb{Z}) \cong (\mathbb{Z}[\tilde{\tilde{D}}_2 \cup \tilde{\tilde{D}}_3])^\perp .$$

A proof of the theorem is done, if we have explicitely determined the divisors D_∞ and K_∞, which will be done in the following TABLEs 5. and 6.. An explicite excecution of the calculations is as described in §5 and is omitted from this paper. For a proof of the Corollary, see (5.9).

The following TABLE 6. describes the dual graph of D_∞ and its decomposition $D_1 \cup D_2 \cup D_3$ for each (a,b,c;h) of TABLE 1. These data together with that of the position of branching points on $E_\infty \cong E_0$ and $A := (p_1, \ldots, p_r)$ in the TABLE 3., completely determine the divisor D_∞ at infinity.

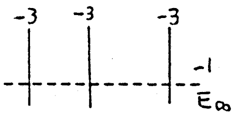
In the following TABLE 5. we summerize the data: $D_1 \cup D_2$ and $\tilde{\tilde{D}}_2$.

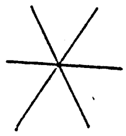
Here D_1 (resp. D_2) is described by dotted (resp. real) lines.

TABLE 5.

$\varepsilon = -1$ The configulation D_1 is always void. The configulation D_2 is a union of smooth rational curves whose intersection diagram is one of the affine Coxeter diagrams of type $\tilde{D}_4$ or $\tilde{E}_k$ (k=6,7,8) (cf TABLE 6.).

$\varepsilon = -2$, $m_- = 3$ and 2.

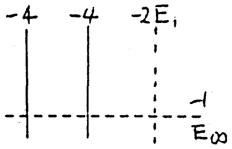
$D_1 \cup D_2 =$ 

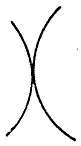
$\xrightarrow{\pi}$ $\pi(D_2) = \tilde{\tilde{D}}_2$  $\tilde{\tilde{D}}_2^2 = 0.$

$K_\infty = E_\infty$.

Three smooth rational curves, intersecting transversaly at a point.

$\varepsilon = -3$, $m_- = 3$ and 2.

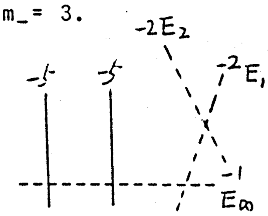
$D_1 \cup D_2 =$ 

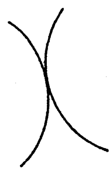
$\xrightarrow{\pi}$ $\pi(D_2) = \tilde{\tilde{D}}_2$  $\tilde{\tilde{D}}_2^2 = 0.$

$K_\infty = 2E_\infty + E_1$

Two smooth rational curves, contacting at a point with order 2.

$\varepsilon = -4$, $m_- = 3$.

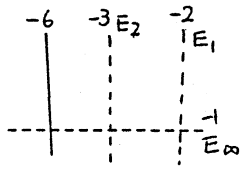
$D_1 \cup D_2 =$ 

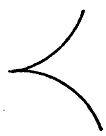
$\xrightarrow{\pi}$ $\pi(D_2) = \tilde{\tilde{D}}_2$  $\tilde{\tilde{D}}_2^2 = 2.$

$K_\infty = 3E_\infty + 2E_1 + E_2$.

Two smooth rational curves, contacting at a point with order 3.

$\varepsilon = -5$, $m_- = 4$, 3 and 2.

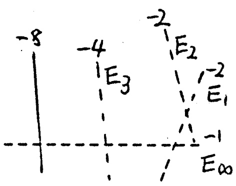
$D_1 \cup D_2 =$ 

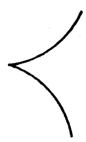
$\xrightarrow{\pi}$ $\pi(D_2) = \tilde{\tilde{D}}_2$  $\tilde{\tilde{D}}_2^2 = 0.$

$K_\infty = 4E_\infty + 2E_1 + E_2$.

A rational curve with a (2,3) cusp.

$\varepsilon = -7$, $m_- = 4$.

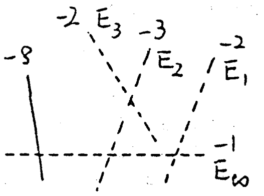
$D_1 \cup D_2 =$ 

$\xrightarrow{\pi}$ $\pi(D_2) = \tilde{\tilde{D}}_2$  $\tilde{\tilde{D}}_2^2 = 4.$

$K_\infty = 6E_\infty + 4E_1 + 2E_2 + E_3$.

A rational curve with a (3,4) cusp.

$\mathcal{E} = -7, \quad m_- = 3.$

$D_1 \cup D_2 =$ 

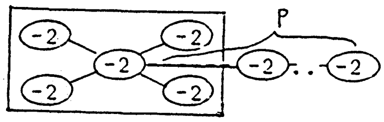
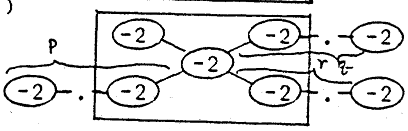
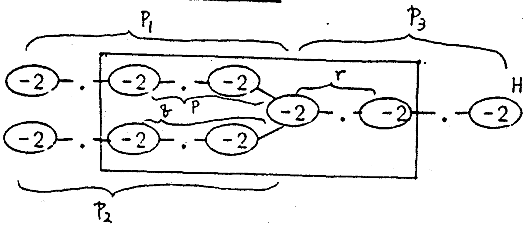
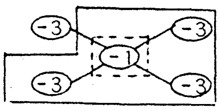
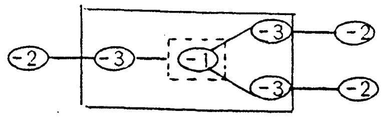
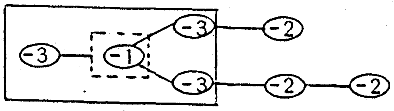
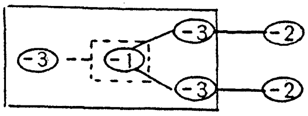
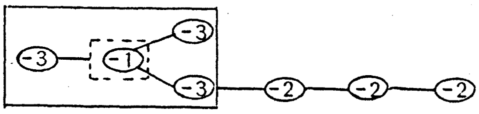
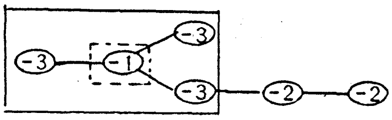
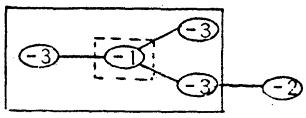
$\longrightarrow \quad \pi(D_2) = \tilde{\tilde{D}}_2$  $\tilde{\tilde{D}}^2 = 4.$

$K_\infty = 6E_\infty + 3E_1 + 2E_2 + E_3$.

A rational curve with a (2,5) cusp.

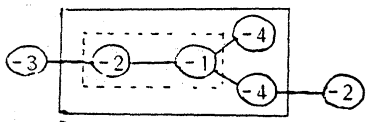
TABLE 6.

(The subdiagrams surounded by a real square describes $D_1 \cup D_2$ of (2.4.1) and the subdiagrams surounded by a dotted square describes D_1 .)

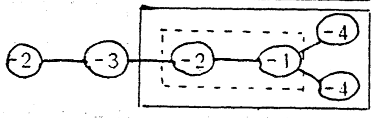
(a,b,c;h)	(m_-, m_0)	dual graph	
$\mathcal{E} = -1$			
(2,2,3;8) (2,2,5;10)	(1,2)		, Here p = 2 or 3.
(2,3,3;9) (2,3,4;10) (2,3,6;12) (2,4,5;12) (2,4,7;14) (2,6,9;18)	(1,1)		, Here (p,q,r) = (3,3,3),(2,3,4), (2,3,3),(2,2,5), (2,2,4),(2,2,3).
14 systems of weights of type II	(1,0)		Here (p,q,r) =(3,3,3(,(2,4,4), or (2,3,6). (p_1, p_2, p_3) = the set of Dolgachev numbers.
$\mathcal{E} = -2$			
(3,3,4;12)	(3,1)		
(3,5,5;15)	(2,0)		
(3,5,7;17)	(2,0)		
(3,5,10;20)	(2,0)		
(3,7,9;21)	(2,0)		
(3,7;12;24)	(2,0)		
(3,10,15;30)	(2,0)		

$\varepsilon = -3$

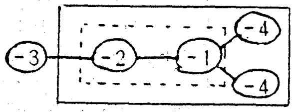
(4,5,7;19) (3,0)



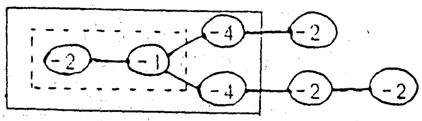
(4,5,8;20) (3,0)



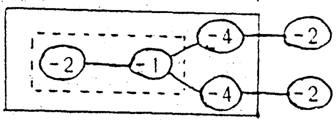
(4,5,12;24) (3,0)



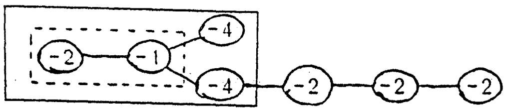
(4,7,10;24) (2,0)



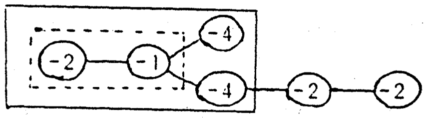
(4,7,14;28) (2,0)



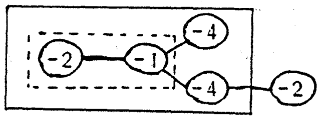
(4,10,13;30) (2,0)



(4,10,17;34) (2,0)

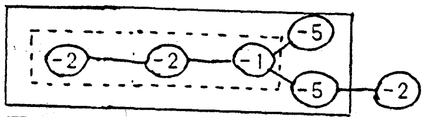


(4,14,21;42) (2,0)

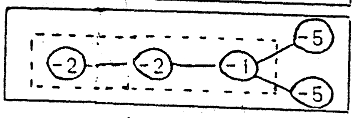


$\varepsilon = -4$

(5,6,9;24) (3,0)

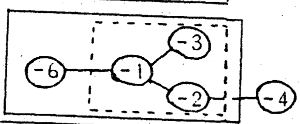


(5,6,15;30) (3,0)

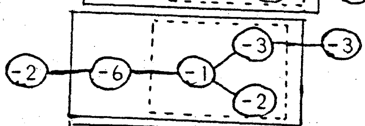


$\varepsilon = -5$

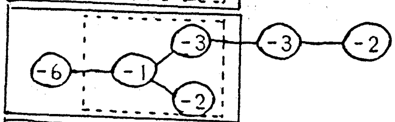
(6,7,9;27) (4,0)



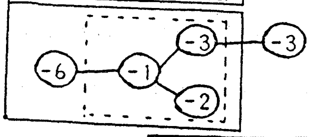
(6,8,11;30) (3,0)



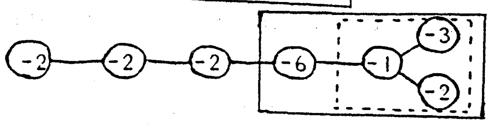
(6,8,13;32) (3,0)



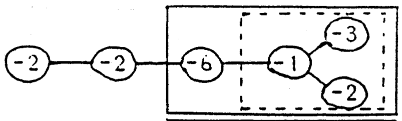
(6,8,19;38) (3,0)



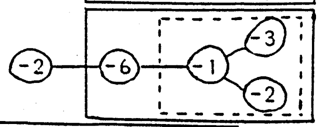
(6,16,21;48) (2,0)



(6,16,27;54) (2,0)

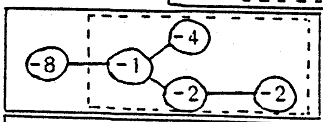


(6,22,33;66) (2,0)

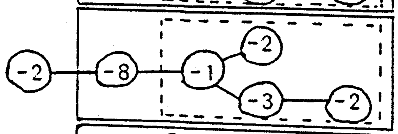


= -7

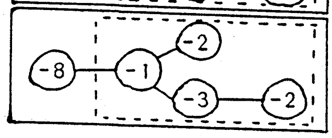
(8,9,12;36) (4,0)



(8,10,15;40) (3,0)



(8,10,25;50) (3,0)



As a consequence of the above explicite description of the divisor D_∞ at infinity, we have the following:

Assertion Except for the case: $m_- = 1$ and $m_0 = 0$ (corresponding to 14 exceptional singularities), the triple (ε, m_-, m_0) determines D_1, D_2 of D_∞.

Note 1. It is quorious to observe that the cannonical divisor and the resolution graph of the singularity X_0 is also determined by the same triple (ε, m_-, m_0) (cf. (2.4) Assertion iv), v) and TABLE 4.). Since these numbers ε, m_- and m_0 are well defined for all Gorenstein singularity with a $\mathbb{C}^*$-action, it may be reasonable to ask the following:

Conjecture Let X_0 be a minimally elliptic singularity with C*-action. Then a smoothing X_t of X_0 over a positively graded part of the parameter, is naturally compactified by a K3 surface, whose structure such as described in (2.4) Assertion iv),v) and (2.5) Assertion deepends only on the triple (ε, m_-, m_0).

Note 2. There are 9 more regular systems of weights with $\varepsilon = -1$ besides those of the TABLE 1. The Milnor fibers are also compactified by K3 surfaces. In 6 cases of them, the divisor D_∞ is a smooth elliptic curves with $D_\infty^2 = 0$. Hence the surface $\tilde{\tilde{X}}_t$ admitts a structure of elliptic fibrations (cf §3).

§ 3 The classs having one negative exponent with 0 exponents

In this paragraph we study surfaces for regular system of weights (a,b,c;h) which has ε as the only negative exponent and 0 as an exponent. If $\varepsilon = -1$ then the corresponding singularities are Fuchsian and hence the corresponding surfaces are K3 as stated in the introduction. Otherwise we shall see that the surfaces are of Kodaira dim 1 with elliptic fibrations over $\mathbb{P}^1$ (see (3.5),(3.6)).

(3.1) System of weights. There are 9+2+1 reduced regular systems of weights which has one negative exponent and some 0 exponents according as $\varepsilon = -1, -2$ or -3, which are listed in the following TABLE 8. (The case $\varepsilon = -1$ is already treated in [23] so that we shall omitt the case from the consideration in this paper.) (Proof. For a system (a,b,c;h) after the smallest exponent ε, the next small exponent is + min(a,b,c). Hence the condition on the sysstem implies ε + min(a,b,c) = 0. Further if $\varepsilon \neq -1$, then 1 must be an exponent for the system (cf (5.5),[24]), which implies $-\varepsilon + 1 \in (a,b,c)$. A calculation similar for the TABLE 1 shows the result.)

TABLE 8.

(a,b,c;h)	exponents
$\varepsilon = -2$	
(2,3,5;12)	-2,0,1,2,3,3,4,4,5,6,6,6,7,8,8,9,9,10,11,12,14
(2,3,7;14)	-2,0,1,2,3,4,4,5,6,6,7,7,8,8,9,10,10,11,12,13,14,16
$\varepsilon = -3$	
(3,4,5;15)	-3,0,1,2,3,4,5,5,6,6,7,8,9,9,10,10,11,12,13,14,15,18

Note that the multiplicity a_0 of zero exponents is 1 in all cases.

(3.2) Polynomial $f(x,y,z,\lambda)$. For each system of weights (a,b,c;h) of the TABLE 8., we associate: i) a weighted homogeneous polynomial $f(x,y,z,\lambda)$ with a parameter λ for the weight (5.5.2), ii) the Milnor number μ and the signature (μ_+, μ_0, μ_-) of the Milnor fiber (5.7.4), iii) the dimentions (m_-, m_0, m_+) of deformation of f (5.7.2).

TABLE 9.

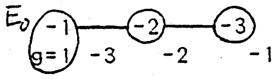
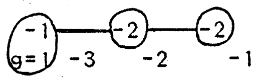
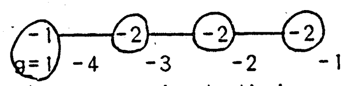
(a,b,c;)	μ	μ_+,μ_0,μ_-	m_- m_0 m_+	polynomial	restriction
(2,3,5;12)	21	2,2,17	3,1,17	$x^6 + y^4 + xz^2 + \lambda x^2yz$	$\lambda^4-64\neq 0$.
(2,3,7;14)	22	2,2,18	3,1,18	$x^7 + xy^4 + z^2 + \lambda x^2yz$	$\lambda^4-64\neq 0$.
(3,4,5;15)	22	2,2,18	4,1,17	$x^5 + xy^3 + z^3 + \lambda x^2yz$	$\lambda^3+27\neq 0$.

Note that the number m_0 of the parameter λ (=dimension of homogeneous deformation of f) is always 1. Another normal form will be given in § 4 TABLE 14..

(3.3) <u>Resolution</u>. The minimal good resolution of the singularity $X_0 := \{(x,y,z)\in\mathbb{C}^3 : f(x,y,z,\lambda)=0\}$ is described in (5.6). Numerically it is determined by the data: the genus $g(E_0)$ and the self-intersection number E_0^2 of the central curve E_0, the set A of the order of cyclic groups and $d := -\varepsilon$.

In the TABLE 10., we give such numerical data and the resolution graph with the coefficients of the cannonical divisor near by for polynomials of TABLE 9..

TABLE 10.

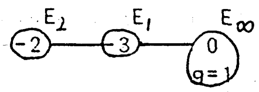
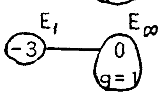
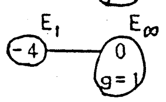
(a,b,c;h)	$g(E_0)$	E_0^2	A	resolution graph
(2,3,5;12)	1	-1	5	
(2,3,7;14)	1	-1	3	
(3,4,5;15)	1	-1	4	

Note. The shape of the dual graph and the cannonical divisor depends only on the triple (ε, m_-, m_0). (Compare (2.4) Assertion iv),v).)

(3.4) <u>The compactification</u>. The unfolding of the polynomial f, the compactifications $\tilde{\tilde{X}}_t$ of their Milnor fiber X_t for $t\in S$ (or S_f) are described in (5.7),(5,8). The surface $\tilde{\tilde{X}}_t$ is a union of the open part $\tilde{X}_t$ (the resolution of the Milnor fiber X_t) and the divisor at infinity D_∞. The cannonical divisor of $\tilde{\tilde{X}}_t$ is a sum $K_\infty + \sum_{x\in X_t} K_x$, where $\mathrm{supp}(K_\infty)\subset D_\infty$ and the second term K_x vanishes away for $t\in S_f$.

In the TABLE 11., we describe the dual graph of D_∞ and the cannonical divisor K_∞.

TABLE 11.

(2,3,5;12)		$K_\infty = E_\infty$, $E_\infty^2 = 0$.
(2,3,7;14)		$K_\infty = E_\infty$, $E_\infty = 0$.
(3,4,5;15)		$K_\infty = 2E_\infty$, $E_\infty^2 = 0$.

In the above table, the vertex in the right terminal of the graphs denotes the curve E_∞, which is an elliptic curve of self intersection zero. Note that the cannonical divisor K_∞ is determined by the triple (ε, m_-, m_0)(Compare (2.5) Assertion.)

(3.5) Now we have the following descriptions of the surface $\tilde{\tilde{X}}_t$ for $t \in S_f$ (cf(5.7)ii)).

i) The surface $\tilde{\tilde{X}}_t$ is minimal.

ii) The geometric genus $P_g(\tilde{\tilde{X}}_t)$ is equal to 1. The second Chern number c_1^2 is equal to 0.

iii) The Kodaira dimension of the surface is equal to 1.

iv) The complete linear system $|-\varepsilon E_\infty|$ defines an elliptic fibration of $\tilde{\tilde{X}}_t$ over $\mathbb{P}^1$ such that $-\varepsilon E_\infty$ is a multiple fiber and E_1 (in the notation of the TABLE 11.) is a $-\varepsilon$-ple section of the fibration. (See (3.6) for details.)

Proof. i) Since K_∞ is an efective elliptic curve, the adunction relation shows that $\tilde{\tilde{X}}_t$ is minimal and that $\tilde{\tilde{X}}_t$ is not a ruled surface.

ii) The first Chern number c_2 = Euler number of X $= 1+\mu+\#$(irreducible components of $D_\infty \setminus E_\infty$) = 24 (TABLE's 9 and 11). The second Chern number $c_1^2 = K_\infty^2 = E_\infty^2 = 0$.

Hence the Noether's formula $P_g + 1 = (c_1^2 + c_2)/12$ implies $P_g = 1$.

iii) $K_\infty^2 = 0$ implies that $\kappa \neq 2$. Since $\tilde{\tilde{X}}_t$ is not ruled, κ is only possible to be 1.

(3.6) As was stated in (3.5) iv), we see in this section that:

The complete linear system $|-\varepsilon E_\infty|$ defines an elliptic fibration of $\tilde{\tilde{X}}_t$ over $\mathbb{P}^1$.

First let us see that the $\ell(-\varepsilon E_\infty) = 2$ and $|-\varepsilon E_\infty|$ is spanned by the constant $1=w/w$ and $x/w^{-\varepsilon}$, where $(x:y:z:w)$ is the homogeneous coordinate for the ambiant weighted projective space $\mathbb{P}(a,b,c,1)$ of $\bar{X}_t$. (Recall that $\tilde{\tilde{X}}_t$ is the resolution of $\bar{X}_t$ and E_∞ is the strict transform of the divisor in $\bar{X}_t$ defined by $w = 0$ (cf (5.8)). Since $\deg(w) = 1$, homogeneous polynomials in (x,y,z,w) of degree less or equal than $-\varepsilon$ is either one of $1, w, \cdots, w^{-\varepsilon}, x$. Hence the complete linear system $|-\varepsilon E_\infty|$ is contained in the space spaned by 1 and $x/w^{-\varepsilon}$. In fact we shall see by explicite calculations of each cases, the function $x/w^{-\varepsilon}$ is holomorphic on the exceptional set of the resolution $\tilde{\tilde{X}}_t \longrightarrow \bar{X}_t$. Before we describe each individual cases, we summerize some generality of the fibration as a statment, which are verryfied by case by case.

i) The rational function $x/w^{-\varepsilon}$ on $\tilde{\tilde{X}}_t$ for $t \in S_f$ defines a flat morphism:

$$\pi = x/w^{-\varepsilon} : \tilde{\tilde{X}}_t \longrightarrow \mathbb{P}^1.$$

ii) The fiber $\pi^{-1}(\infty)$ is $-\varepsilon E_{\infty}$.

iii) The restriction of π on the curve $E_1 \subset \tilde{\bar{X}}_t$ ((cf TABL II)) defines a $-\varepsilon$-fold covering of $\mathbb{P}^1$ which is branching at ∞ of order $-\varepsilon$ and at some other points.

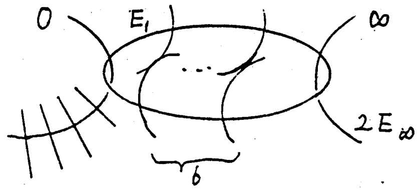
iv) The general fibers of π are elliptic curves.

v) In the following we figure the singular fibers of the fibrations $\pi : \bar{X}_t \longrightarrow \mathbb{P}^1$ for $t \in S \cap (0 \times \mathbb{C}^{m_c} \times 0)$ = the degree zero part of the parameterspace S.

(2,3,5;12) equation: $X^6 + Y^4 + XZ^2 + \lambda X^2YZ - W^{12} = 0.$

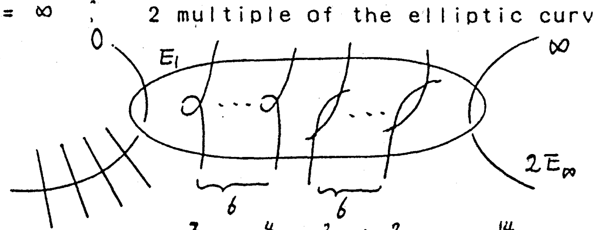
case $\lambda = 0$

location		singular fiber
$x/w^2 = 0$	:	a union of 5 smooth rational curves, intersecting in $\tilde{D}_4$ diagram.
$(x/w^2)^6 = 1$	:	two smooth rational curve contacting at a point.
$x/w^2 = \infty$	:	2 multiple of the elliptic curve E_∞.



case $\lambda \neq 0$

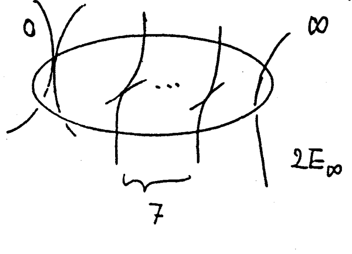
location		singular fiber
$x/w = 0$	:	a union of 5 smooth rational curves, intersecting in $\tilde{D}_4$ diagram.
$(x/w^2)^6 = 1$	:	a rational curve with a node.
$(1-\frac{\lambda^4}{64})(x/w^2)^6 = 1$	:	two smooth rational curve crossing at two points.
$x/w = \infty$	:	2 multiple of the elliptic curve E_∞.



(2,3,7;14) equation: $X^7 + XY^4 + Z^2 + \lambda X^2YZ - W^{14} = 0.$

case $\lambda = 0$

location		singular fiber
$x/w^2 = 0$	:	two smooth rational curves contacting at 0 on E_1.
$(x/w^2)^7 = 1$	:	two smooth rational curves contacting at a point.
$x/w^2 = \infty$	:	2 multiple of the elliptic curve E_∞.



case $\lambda \neq 0$

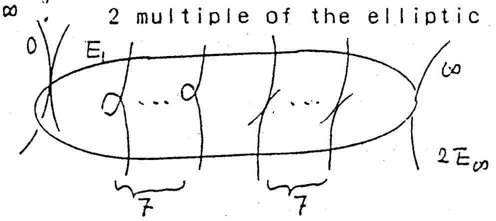
location ; singular fiber

$x/w = 0$; two smooth rational curves contacting at 0 on E_1.

$(x/w\ ) = 1$; a rational curve with a node.

$(1-\frac{\lambda^4}{64})(x/w\ ) = 1$; two smooth rational curve crossing at two points.

$x/w = \infty$; 2 multiple of the elliptic curve E_∞.



(3,4,5;15) equation: $X^5 + XY^3 + Z^3 + \lambda X^2YZ - W^{15} = 0$.

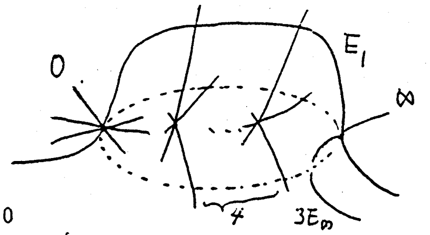
case $\lambda = 0$

location ; singular fiber

$x/w^3 = 0$; three smooth rational curves crossing at 0 on E_1.

$(x/w^3)^5 = 1$; three smooth rational curves crossing at a point.

$x/w^3 = \infty$; 2 multiple of the elliptic curve E_∞.



case $\lambda \neq 0$

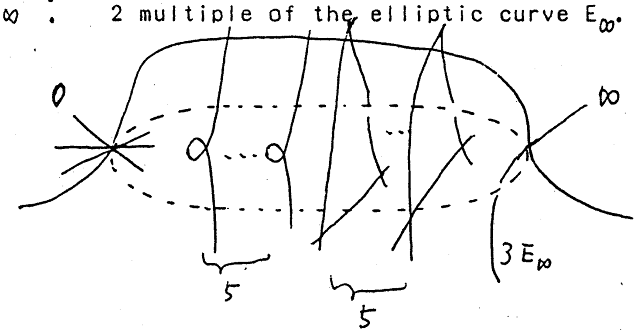
location ; singular fiber

$x/w^3 = 0$; three smooth rational curves crossing at 0 on E_1.

$(x/w^3)^5 = 1$; a rational curve with a node.

$(1-\frac{\lambda^3}{27})(x/w^3)^5 = 1$; three smooth rational curve forming a triangle.

$x/w^3 = \infty$; 2 multiple of the elliptic curve E_∞.



§4. The class for the smallest exponent ε eaquals to -2

In this paragraph we study surfaces for regular system of weights (a,b,c;h) such that $\varepsilon := a+b+c-h = -2$. According as the multiplicity a_0 of zero exponent is 0, 1 or >1, the surface is K3, of Kodaira dim 1 or general type (see (4.5)).

(4.1) In the TABLE 13., we list up reduced regular system of weights with $\varepsilon = -2$. (Due to the general inequality $-\varepsilon+1 \geq \min(a,b,c)$ (cf (5.5.7),[24]), we have only three cases min(a,b,c)= 1,2 or 3. Detailed calculations are cumbersome and omitted.)

According to the multiplicities of exponents, they are divided into groups.

TABLE 13.

(a,b,c;h)	exponents
(3,10,15;30)	-2,1,4,7,8,10,11,13,14,16,17,19,20,22,23,26,29,32
(3,7,12;24)	-2,1,4,5,7,8,10,11,12,13,14,16,17,19,20,23,26
(3,7,9;21)	-2,1,4,5,7,7,8,10,11,13,14,14,16,17,20,23
(3,5,10;20)	-2,1,3,4,6,7,8,9,10,11,12,13,14,16,17,19,22
(3,5,7;17)	-2,1,3,4,5,6,7,8,9,10,11,12,13,14,16,19
(3,5,5;15)	-2,1,3*2,4,6*2,7,8,9*2,11,12*2,14,17
(3,3,4;12)	-2,1*2,2,4*3,5,5,7,7,8*3,10,11*2,14
(2,3,7;14)	-2,0,1,2,3,4*2,5,6*2,7*2,8*2,9,10*2,11,12,13,14,16
(2,3,5;12)	-2,0,1,2,3*2,4*2,5,6*3,7,8*2,9*2,10,11,12,14
(1,6,9;18)	-2,-1,0,1,2,3,4*2,5*2,6*2,7*2,8*2,9*2,10*2,11*2,12*2,13*2,14*2,15,16,17, 18,19,20
(1,5,8;16)	-2,-1,0,1,2,3,3,4*2,5*2,6*2,7*2,8*3,9*2,10*2,11*2,12*2,13*2,14,15,16,17, 18
(1,5,7;15)	-2,-1,0,1,2,3*2,4*2,5*3,6*2,7*2,8*2,9*2,10*3,11*2,12*2,13,14,15,16,17
(1,3,6;12)	-2,-1,0,1*2,2*2,3*2,4*3,5*3,6*3,7*3,8*3,9*2,10*2,11*2,12,13,14
(1,3,5;11)	-2,-1,0,1*2,2*2,3*3,4*3,5*3,6*3,7*3,8*3,9*2,10*2,11,12,13
(1,3,3;9)	-2,-1,0,1*3,2*3,3*3,4*4,5*4,6*3,7*3,8*3,9,10,11
(1,2,5;10)	-2,-1,0*2,1*2,2*3,3*3,4*4,5*4,6*4,7*3,8*3,9*2,10*2,11,12
(1,2,3;8)	-2,-1,0*2,1*3,2*4,3*4,4*5,5*4,6*4,7*3,8*2,9,10
(1,1,4;8)	-2,-1*2,0*3,1*4,2*5,3*6,4*7,5*6,6*5,7*4,8*3,9*2,10
(1,1,3;7)	-2,-1*2,0*3,1*5,2*6,3*7,4*7,5*6,6*5,7*3,8*2,9

(1,1,2;6) -2,-1*2,0*4,1*6,2*8,3*8,4*8,5*6,6*4,7*2,8

(1,1,1;5) -2,-1*3,0*6,1*10,2*12,3*12,4*10,5*6,6*3,7

Here recall the convention that u*v means u,...,u (v-copies).

(4.2) The polynomial $f(x,y,z,\lambda)$, (m_-, m_0, m_+) and (μ_+, μ_0, μ_-)

Let (a,b,c;h) be a system of weights of TABLE 13. In the TABLE 14., we shall give a weighted homogenous polynomial $f(x,y,z,\lambda)$ with m_0-number of parameters for the weights, where m_+, m_0 and m_- are the numbers of parameters of an uinversal unfolding of f with positive, zero and negative weights respectively(5.7.2).

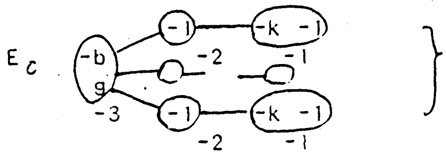
The first 7 systems of TABLE 8. is already treated in TABLE 2, and are omitted.

TABLE 14.

(a,b,c;h)	μ	$\mu_+ \mu_0 \mu_-$	m_- m_0 m_+	polynomial	
(2,3,7;14)	22	2,2,18	3,1,18	$x(x^3-y^2)(x^3-\lambda y^2)+z^2$	$\lambda \neq 0,1$
(2,3,5,12)	21	2,2,17	3,1,17	$(x^3-y^2)(x^3-\lambda y^2)+z^2x$	$\lambda \neq 0,1$
(1,6,9;18)	34	4,2,28	4,1,29	$y(x^6-y)(x^6-\lambda y)+z^2$	$\lambda \neq 0,1$
(1,5,8;16)	33	4,2,27	4,1,28	$xy(x^5-y)(x^5-\lambda y)+z^2$	$\lambda \neq 0,1$
(1,5,7;15)	32	4,2,26	4,1,27	$y(x^5-y)(x^5-\lambda y)+xz^2$	$\lambda \neq 0,1$
(1,3,6;12)	33	4,2,27	5,2,26	$y(x^3-y)(x^3-\lambda_1 y)(x^3-\lambda_2 y)+z^2$	$\lambda_i \neq \lambda_j$, $\lambda_i \neq 0,1$
(1,3,5;11)	32	4,2,26	5,2,25	$x^2y(x^3-\lambda_1 y)(x^3-\lambda_2 y)+y^2z+xz^2$	$\lambda_i \neq \lambda_j$, $\lambda_i \neq 0,1$
(1,3,3;9)	32	4,2,26	6,3,23	x y + y + z + (y + yz + z)x	
(1,2,5;10)	36	4,4,28	6,3,27	$y(x^2-y)(x^2-\lambda_1 y)(x^2-\lambda_2 y)(x^2-\lambda_3 y)+z^2$	$\lambda_i \neq \lambda_j$, $\lambda_i \neq 0,1$
(1,2,3;8)	35	4,4,27	7,4,24	$x^5z+\prod_{i=1}^{4}(y-\lambda_i x^2)+z^2y$	$\lambda_i \neq \lambda_j$, $\lambda_i \neq 0,1$
(1,1,4;8)	49	6,6,37	10,5,34	$xy(x-y)(x-\lambda_1 y)\dots(x-\lambda_5 y)+z^2$	$\lambda_i \neq \lambda_j$, $\lambda_i \neq 0,1$
(1,1,3;7)	48	6,6,36	11,6,31	$z^2x+g(x,y)z+h(x,y)$ where g,h are homogeneous of degree 4,7 respectively.	
(1,1,2;6)	50	6,8,36	13,8,29	$z^3+g(x,y)z+h(x,y)$ where g,h are homogeneous of degree 4,6 respectively.	
(1,1,1;5)	64	8,12,44	20,12,32	f(x,y,z): homogeneous of degree 5	

(4.3) Resolution. The minimal good resolution of the singularity $X_0 := \{(x,y,z)\in\mathbb{C}^3 : f(x,y,z,\lambda)=0\}$ is described in (5.6). It is numerically deteremined by the data: the genus $g(E_0)$ and the self intersection number E_0^2 of the central curve E_0, the set A $(p_1,\dots,p_r)$ of the order of cyclic groups and $\alpha := -\varepsilon = 2$. (See TABLE 15.)

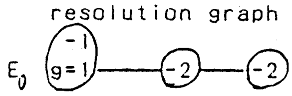
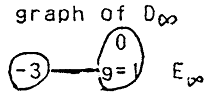
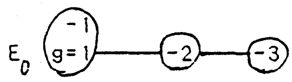
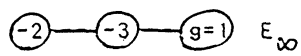
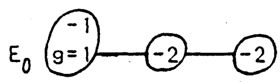
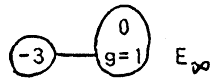
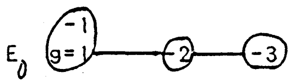
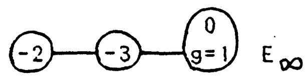
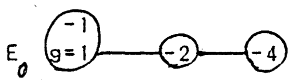
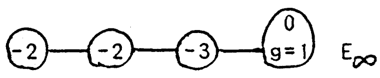
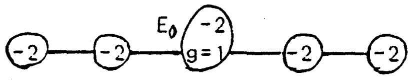
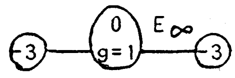
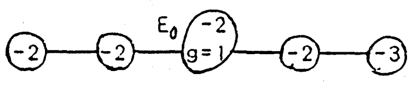
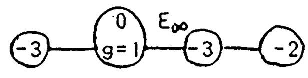
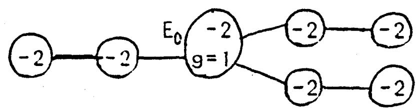
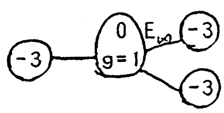
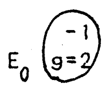
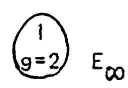
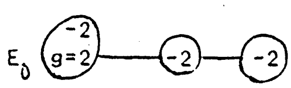
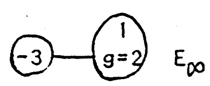
For a system (a,b,c;h) of TABLE 13., the set A consists of odd integers due to (5.6.5). Hence the dual graph for the minimal good resolution of the singularity and the coefficients of the cannonical divisor K_0 of the singularity are as follows:



r-branches,

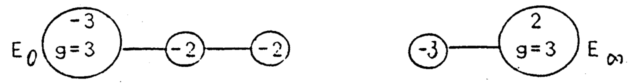
where $k_i := (p_i - 1)/2$ $(i = 1,\dots,r)$, $b := -E_0^2 = 1 + a_1 - a_0$, and $g :=$ genus$(E_0) = 1+b-r$.

TABLE 15.

(a,b,c;h)	A	resolution graph	dual graph of D_∞
(2,3,7;14)	3		
(2,3,5;12)	5		
(1,6,9;18)	3		
(1,5,8;16)	5		
(1,5,7,15)	7		
(1,3,6;12)	3,3		
(1,3,5;11)	3,5		
(1,3,3;9)	3,3,3		
(1,2,5;10)			
(1,2,3;8)	3		

(1,1,4;8)

(1,1,3;7) 3



(1,1,2;6)

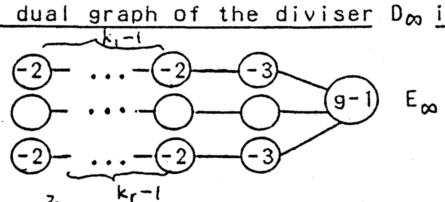
(1,1,1;5)

Note. The shape of the dual graph and the coefficients of cannonical divisor are determined by the triple (ε, m_-, m_0) except for the pair (1,3,3;9) and (1,2,5;10), which are already distinguished by a_0(=the multiplicity of zero exp.)(cf (2.4) Ass.).

(4.4) Compactifications. The compactifications $\tilde{\bar{X}}_t$ of the Milnor fiber X_t $(t \in S_{cr} S_f)$ are described in (5.8). $\tilde{\bar{X}}_t$ is a union $\bar{X}_t \cup D_\infty$ of the resolution $\tilde{X}_t$ of the Milnor fiber and the divisor D_∞ at infinity. The cannonical divisor of $\tilde{\bar{X}}_t$ is a sum $K_\infty + \sum_{x \in X_t} K_x$ such that $\mathrm{supp}(K_\infty) \subset D_\infty$ and the second term K_x is zero for $t \in S_f$.

Let us describe more details for the case of $\varepsilon = -2$.

Assertion i) The dual graph of the diviser D_∞ is the following (See TABLE 15.):

r-branches 

, where $k_i := (p_i - 1)/2$ for $p_i \in A$.

ii) $K_\infty = E_\infty$ and $K_\infty^2 = E_\infty^2 = g-1$, where $g := g(E_\infty) = g(E_0) = a_0$.

Proof. i) Since $p \in A$ is an odd integer, it has the following continued fraction:

$$p/(p-2) = 2 - \cfrac{1}{\underbrace{2 - \cfrac{1}{2 \cdots - \cfrac{1}{2}}}_{k-1} - \cfrac{1}{3}}, \quad \text{where } p = 2k+1.$$

This gives the intersection numbers for the curves on the branches of D_∞.

ii) Let us put $K_\infty = E_\infty + K'$, where K' is a diviser with support on the branches. The adjunction formula $K_\infty E + E = 2g(E) - 2$ implies that $K'E = 0$ for all curves E on the branches of D_∞. Since the intersection matrixes on branches are nondegenerate, $K' = 0$ and hence $K_\infty = E_\infty$. Again applying the adjunction formula $2g(E_\infty) - 2 = K_\infty E_\infty + E_\infty$, we obtain ii). QED

(4.5) Summerizing those calculations above, the surfaces $\tilde{\tilde{X}}_t$ ($t \in S_f$) are as follows.

We distinguish three cases according to $a_0 - 1 = g(E_0) - 1$.

I. $g(E_\infty) - 1 < 0$.

In this case $K_\infty = E_\infty$ is an exceptional curve of the first kind. The cannonical bundle of the blown down surface $\overset{\approx}{X}_t = \tilde{\tilde{X}}_t / E_\infty$ is trivial for $t \in S_f$, so that $\overset{\approx}{X}_t$ is a K3 surface with a configulation of three lines crossing normally at a point.

This case is already studied in §3 , so that we omitt further details.

II. $g(E_\infty) - 1 = 0$.

In this case $K_\infty = E_\infty$ is a smooth elliptic curve with self-intersection zero and hence the surface is minimal. $\tilde{\tilde{X}}_t$ for $t \in S_f$ is of Kodaira dimension 1, which has a structure of elliptic fibration over $\mathbb{P}^1$ with E_∞ as a regular fiber.

(That K_∞ is an elliptic curve implies $\tilde{\tilde{X}}_t$ is minimal. Then $K_\infty^2 = 0$ implies that the Kodaira dimension of $\tilde{\tilde{X}}_t$ can not be 2. Since $\tilde{\tilde{X}}_t$ can not be a ruled surface (K_∞ is effective), the Kodaira dimension of $\tilde{\tilde{X}}_t$ is only possible to be 1. The fact the irregularity q of the surface is zero (5.9) implies that $\tilde{\tilde{X}}_t$ has a structure of an elliptic fibration over $\mathbb{P}^1$ according to the classification of surfaces [ ]. qed)

An explicite description of the elliptic fibration is given in (4.8).

III) $g(E_\infty) - 1 > 0$.

In this case $K_\infty = E_\infty$ is a smooth curve of genus > 1 , whose selfintersection number $K_\infty^2 = g(E_\infty) - 1$ is positive.

The surface $\tilde{\tilde{X}}_t$ for $t \in S_f$ is minimal and of general type, which satisfy the numerical equality: $P_g = [\, c_1^2/2 \,] + 2$ where P_g is the geometric genus and c_1^2 is the second Chern number of the surface (cf (4.6.2)). For this class of the surface, we refered [ ],[ ].

(For the same reasons as II, $\tilde{\tilde{X}}_t$ is minimal and cannot be ruled. Then the positivi -ty $K_\infty^2 > 0$ implies that $\tilde{\tilde{X}}_t$ is of general type due to classification of surfaces [ ].)

The numerical invariants P_g, c_1^2 and c_2 of the surface $\tilde{\tilde{X}}_t$ is calculated in (4.6).

(4.6) We calculate: the first Chern number c_2, the second Chern number $c_1^2 = K_\infty^2$ and the geometric genus $P_g := h^2(\mathcal{O}_{\tilde{\tilde{X}}})$ for the surfaces $\tilde{\tilde{X}}_t$ ($t \in S_f$). They are easily calculated by the following formula with the data in TABLE's 14,15,16.

$c_2 :=$ Euler # for $\tilde{\tilde{X}}_t$ = (Euler # for $\check{X}_t$) + (Euler # for D_∞)

$= (1 + \mu) + (2 - 2g + \#(\text{irreducible components of } D - E))$.

$c_1^2 := K_\infty^2 = g - 1$.

$P_g + 1 = (c_1^2 + c_2)/12$ (Noether's formula).

The following TABLE 17. gives the invariants of the surfaces and the number of the weight (a,b,c) which is equal to 1 for an application in (4.7).

TABLE 17.

system of weights	c_2	c_1^2	P_g	$\#(e \in (a,b,c):\ e=1)$
(2,3,7;14)	24	0	1	0
(2,3,5;12)	24	0	1	0
(1,6,9;18)	36	0	2	1
(1,5,8;16)	36	0	2	1
(1,5,7;15)	36	0	2	1
(1,3,6;12)	36	0	2	1
(1,3,5;11)	36	0	2	1
(1,3,3;9)	36	0	2	1
(1,2,5;10)	35	1	2	1
(1,2,3;8)	35	1	2	1
(1,1,4;8)	46	2	3	2
(1,1,3;7)	46	2	3	2
(1,1,2;6)	45	3	3	2
(1,1,1;5)	55	5	4	3

As a consequence of the above table, we get the following formula.

(4. 6. 1) $\qquad P_g(\tilde{\tilde{X}}_t) = 1 + \#(\ e\ (a,b,c):\ e=1\ )\quad$ for $t \in S_f$.

Another consequence of the table is the following equality:

(4.6.2) $\qquad P_g(\tilde{\tilde{X}}_t) = [\ c_1^2/2] + 2$

for the last group of 7 systems of weights satisfying the condition $a_0 > 1$.

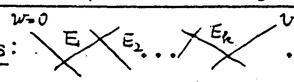
(4.7) The cannonical linear system $|K_\infty|$ for the surfaces $\tilde{\tilde{X}}_t$ ($t \in S_f$) are as follows.

<u>Assertion</u> <u>The module for the linear system</u> $|K_\infty|$ <u>is spanned by w and by the coordinate</u> (x, y, z) <u>such that the coresponding weight</u> $\in$ (a, b, c) <u>is equal to 1.</u>

Proof Recalling $K_\infty = E_\infty$, we have $P_g = h^2(\mathcal{O}_{\tilde{X}}) = h^0(\mathcal{O}(E_\infty)) =$ dim(the space of meromorphic function on $\tilde{\tilde{X}}_t$ which may have at most a simple pole along E_∞.).

Let us show that if the weight (a,b,c) of a coordinate (x,y,z), say x, is 1, then the meromorphic function x/w belongs to the space $H^0(\tilde{\tilde{X}}_t, \mathcal{O}(E_\infty))$. In view of the equality (4.6.1), this proves the assertion. ($\tilde{\tilde{X}}_t$ is not linear.)

First recall that $\tilde{\tilde{X}}_t$ is a resolution of the surface $\bar{X}_t$ in $\mathbb{P}(a,b,c,1)$ by blowing up the cyclic quotient singularities on $\tilde{\tilde{X}}_t$, which appear at the coordinate axis $L_x \cup L_y \cup L_z$ in $\mathbb{P}(a,b,c) := (w=0) \subset \mathbb{P}(a,b,c,1)$. Since E_∞ is the strict transform of the curve $\bar{X}_t \cap \mathbb{P}(a,b,c)$ and hence x/w has simple pole along E_∞, we have only to show that x/w does not have poles on the exceptional set of the resolution $\tilde{\tilde{X}}_t \longrightarrow \bar{X}_t$. The assumption on the weight $a=1$ and the description of the points $\bar{X}_t \cap (L_x \cup L_y \cup L_z)$ (5.6.5) implies the singlar points of $\bar{X}_t$ lie only on L_x. If, for instance, $z \neq 0$ at a singular point, $\bar{X}_t$ is locally at the point a quotient of smooth $Y := \{(x,y,w) \in \mathbb{C}^3 : f(x,y,1) = w^h\}$ by the action of $\zeta \in \mathbb{Z}_p$, $(x,y,w) \longmapsto (\zeta x, \zeta^b y, \zeta w)$. Let (v,w) be a local coordinate system of Y at the fixed point, on which the action of $\zeta \in \mathbb{Z}$ is $(\zeta^\varepsilon v, \zeta w)$ (cf). Let us develop x into a power series $\sum_{i,j} a_{ij} v^i w^j$ in the local coordinates. Sicne $\zeta \in \mathbb{Z}_p$ acts on x as ζx, the power series is a sum over the indixes $(i,j) \in \mathbb{N}_0^2$ such that $-2i + j = 1 \bmod(p)$. In case $j = 0$, the condition $2i + 1 = 0 \bmod(p)$ implies $i = (p-1)/2 + n\,p$ for some $n \in \mathbb{N}_0$. If we have shown that $v^{(p-1)/2}/w$ is holomorphic on the exceptional set of the resolution of the quotient singularty, we have also shown that x/w is holomorphic on the exceptional set. Let us give a sharper form for a later use.

**) <u>Let $E_1, \ldots, E_k$ be the exceptional set for the minimal resolution of the cyclic quotient singularity of the type</u> (p,-2) <u>with</u> $k=(p-1)/2$, <u>which are intersecting as</u>: 

. <u>Then the rational function</u> v^k/w <u>defnes a pole along</u> $w = 0$, <u>a rational parametrization of</u> E_1, <u>and a zero function on</u> $E_2 \cup \ldots \cup E_k$. (Proof omitted)

This complete a proof of the assertion. qed

(4.8) We shall describe the cannonical map $\tilde{\tilde{X}}_t \longrightarrow \mathbb{P}^1$, for each systems of weights.

The details of the calculations are omitted.

(2,3,7;14), (2,3,5;12)

$P_g(\tilde{\tilde{X}}_t) = 1$ for these two cases. Hence the cannonical maps are constants.

(Note that the multiple $-\varepsilon K_{\tilde{\tilde{X}}}$ defines elliptic fibration (3.6).)

(1,6,9;18), (1,5,8;16), (1,5,7;15), (1,3,6;12), (1,3,5;11), (1,3,3;9)

$P_g(\tilde{\tilde{X}}_t) = 2$ and $H^0(\tilde{\tilde{X}}_t, \mathcal{O}(K_\infty)) = [1, x/w]$ for these 6 cases. The cannonical map $\pi = (x;w)\colon \tilde{\tilde{X}}_t \longrightarrow \mathbb{P}^1$ defines an elliptic fibration of $\tilde{\tilde{X}}_t$ as follows:

i) The map π is a flat morphism.

ii) $\pi^{-1}(\infty) = E_\infty$.

iii) The -3 curves of D_∞ (in the TABLE 15) are global sections of the map π.

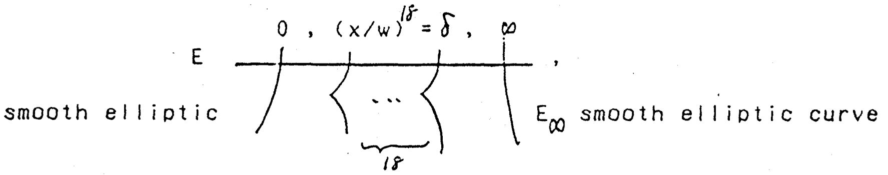
iv) The general fiber of π is a smooth elliptic curve.

v) Singular fibers for $t \in S \cap (0 \times \mathbb{C}^{m_0} \times 0)$ = (the degree 0 subsapce of S) are follows.

(1,6,9;18) equation: $Y(X^6 - Y)(X^6 - \lambda Y) + Z^2 - W^{18} = 0$.

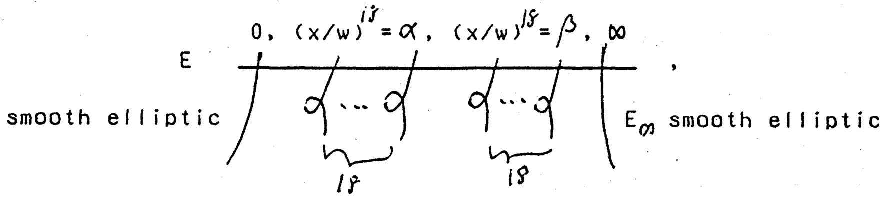
case $\lambda^2 - \lambda + 1 = 0$

location		fiber
$x/w = 0$	:	smooth elliptic curve.
$(x/w)^{18} = \gamma$	:	a rational curve with a (2,3)-cusp.
$x/w = \infty$	:	E_∞ (a smooth elliptic curve)



case $\lambda^2 - \lambda + 1 \neq 0$

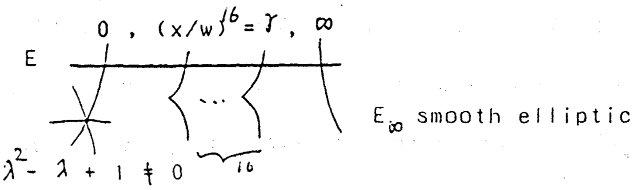
location		fiber
$x/w = 0$	:	smooth elliptic curve.
$(x/w)^{18} = \alpha$	:	a rational curve with a node.
$(x/w)^{18} = \beta$	:	a rational curve with a node.
$x/w = \infty$	:	E (a smooth elliptic curve)



(1,5,8;16) equation: $XY(X^5 - Y)(X^5 - \lambda Y) + Z^2 - W^{16} = 0.$

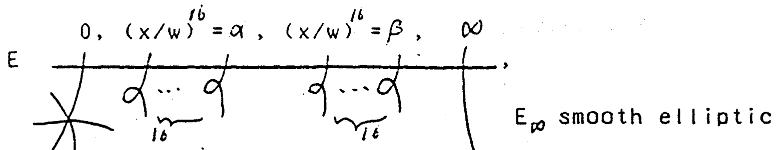
case $\quad - \quad + 1 = 0$

location		fiber
$x/w = 0$	;	a union of 3 smooth rational curves intersecting at a pont.
$(x/w)^{16} = \gamma$	;	a rational curve with a (2,3)-cusp.
$x/w = \infty$	;	E_∞ (a smooth elliptic curve)



case $\lambda^2 - \lambda + 1 \neq 0$

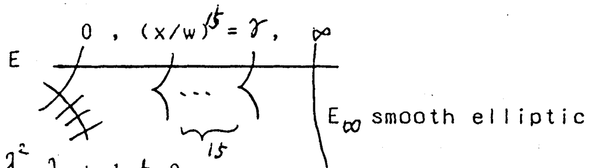
location	fiber
$x/w = 0$	a union of 3 smooth rational curves intersecting at a point.
$(x/w)^{16} = \alpha$	a rational curve with a node.
$(x/w)^{16} = \beta$	a rational curve with a node.
$x/w =$	E (a smooth elliptic curve)



(1,5,7;15) equation: $Y(X^5 - Y)(X^5 - \lambda Y) + XZ^2 - W^{15} = 0.$

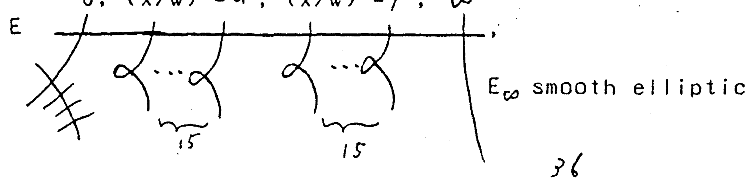
case $\lambda^2 - \lambda + 1 = 0$

location		fiber
$x/w = 0$	:	a union of 5 smooth rational curves intersecting in D .
$(x/w)^{15} = \gamma$	:	a rational curve with a (2,3)-cusp.
$x/w = \infty$	:	E (a smooth elliptic curve)



case $\lambda^2 - \lambda + 1 \neq 0$

location		fiber
$x/w = 0$	:	a union of 5 smooth rational curves intersecting in D .
$(x/w)^{15} = \alpha$	:	a rational curve with a node.
$(x/w)^{15} = \beta$	:	a rational curve with a node.
$x/w = \infty$	:	E_∞ (a smooth elliptic curve)



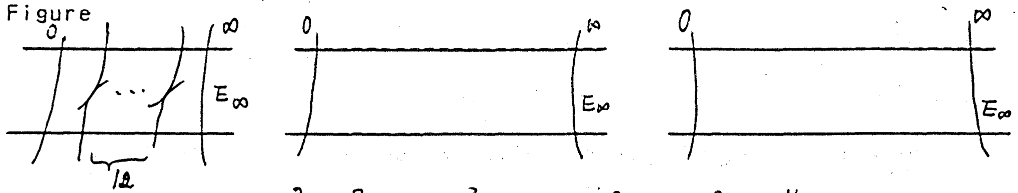
(1,3,6;12) equation: $Y(X^3-\lambda_1 Y)(X^3-\lambda_2 Y)(X-\lambda_1 Y)+Z^2-W^{12}=0.$

case location	fiber
$x/w=0$	a smooth elliptic curves.
$(x/w)^{12}=1$	two smooth rational curves contacting at a pont.
$x/w=\infty$	E_∞ (= a smooth elliptic curve).

case location	fiber
$x/w=0$	a smooth elliptic curves.
$(x/w)^{12}=$	
$(x/w)^{12}=$	
$x/w=\infty$	E_∞ (= a smooth elliptic curve).

case location	fiber
$x/w=0$	a smooth elliptic curves.
$(x/w)^{12}=$	
$(x/w)^{12}=$	
$(x/w)^{12}=$	
$x/w=\infty$	E_∞ (= a smooth elliptic curve).

Figure



(1,3,5;11) equation: $X^2Y(X^3-\lambda_1 Y)(X^3-\lambda_2 Y)+Y^2Z+XZ^2-W^{11}=0.$

case location	fiber
$x/w=0$	two smooth rational curves contacting at a point.
$(x/w)^{11}=1$	two smooth rational curves contacting at a pont.
$x/w=\infty$	E (= a smooth elliptic curve).

case location	fiber
$x/w=0$	two smooth rational curves contacting at a point.
$(x/w)^{11}=$	
$(x/w)^{11}=$	
$x/w=\infty$	E_∞ (= a smooth elliptic curve).

case
location fiber

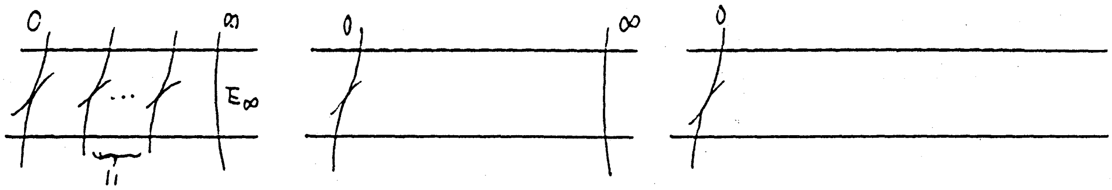
$x/w = 0$ a smooth elliptic curves.

$(x/w) =$

$(x/w) =$

$(x/w) =$

$x/w =$ E (= a smooth elliptic curve).



(1,3,3;9) equation: $XY(X - Y)(X - Y)(X - Y) + Z - W = 0$.

case
location fiber

$x/w = 0$ a smooth elliptic curves.

$(x/w)^9 = 1$ three smooth rational curves crossing at a pont.

$x/w = \infty$ E_∞ (= a smooth elliptic curve).

case
location fiber

$x/w = 0$ a smooth elliptic curves.

$(x/w)^9 =$

$(x/w)^9 =$

$x/w = \infty$ E_∞ (= a smooth elliptic curve).

case
location fiber

$x/w = 0$ a smooth elliptic curves.

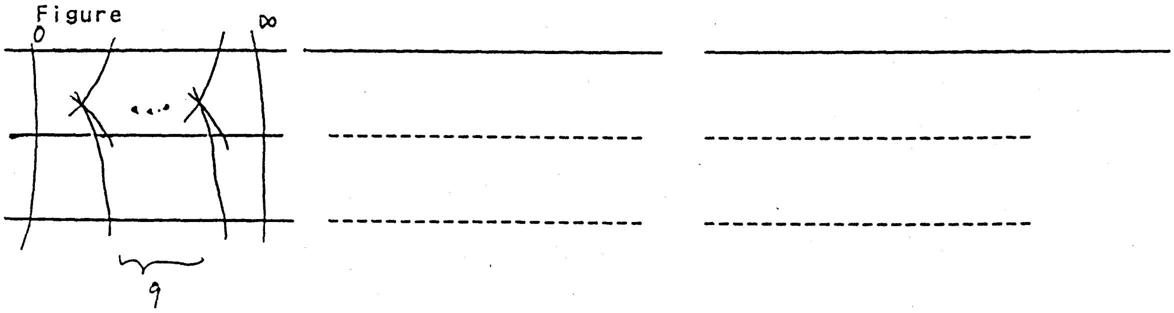
$(x/w)^9 =$

$(x/w)^9 =$

$(x/w)^9 =$

$x/w = \infty$ E (= a smooth elliptic curve).

Figure



(1,2,5;10),(1,2,3;8)

P (X) = 2 for these two cases. The linear system |K | has a fixed point on E . By blowing up X --> X at that fixed point, whose exceptional set will refered as E, we obtain a fibration of :X --> P . The general fiber of is a genus 2 curve and the exceptional set E is a global section. The singular fibers for the special point is as follows

(1,2,5;10)

(1,2,3;8)

(1,1,4;8),(1,1,3;7),(1,1,2;6)

P (X) = 3 for these 3 cases and H (X , (K)) [1,x/w,y/w]. The cannonial map (x,y,w): X --> P defines a covering, whose degree and descriminant are as follows:

(1,1,4;8) equation: Z + g(X,Y,W) = 0 ,where g is homogeneous of degree 8.

is a double covering branching along g = 0.

The discriminant := -4g is homogenous of degree 8.

(1,1,3;7) equation: XZ + g(X,Y,W)Z + h(X,Y,W) = 0, where g and h arehomogenous of degree 4 and 7 respectively.

is a double covering of P branching along a degree 8 curve.

The discriminant := g - 4xh is homogenous of degree 8.

(1,1,2;6) equation: Z + g(X,Y,W)Z + h(X,Y,W) = 0, where g and h are homogeneous of degree 4 and 6 respectively.

is a triple covering of P branching along a degree 12 curve.

The discriminant := h - g is homogenous of degree 12.

(1,1,1;5) equation f(X,Y,Z,W) = 0, where f is homogenous of degree 5.

P (X) = 4 and H (X , (K)) [1,x/w,y/w,z/w] for this case. The cannonical map (x:y:z:w): X --> P defines an embedding of X as a quintic surface in P .

§5 Weighted homogenous singularity of dimension two

(5.1) This § is a review on the weighted homogeneous singularities of dimension two, studied by V.I.Dolgachev, E.Looijenga, P.Orlik, H.Pinkham, P.Wagreich, J. Wahl and the auther. We describe uniformization, resolution, comapctification of Milnor fibers for mainly hypersurface cases in connection with regular system of weights to fix notations for §'s 2,3 and 4. Many of the results are well-known or elementary so that we give only references or sketchy proofs.

(5.2) Cyclic extensions of PSL(2,$\mathbb{R}$) and their action on $\mathbb{H}_d$.

In the following, we present a weighted homogeneous singularity X_0 as a quotient variety by a splitting factor for a cyclic extention of a Fuchsian group (5.4.1). This is a reformulation of a presentation of a quasi-homogneous singularity by a use of automorphic forms by Dolgachev [7], Wagreich [35].

i) Let $\mathbb{H} := \{z\in\mathbb{C} : \mathrm{Im}(z) > 0\}$ be the comlex upper half plane. As usual $\mathrm{Aut}(\mathbb{H})$ is isomorphic to $PSL(2,R)=SL(2,\mathbb{R})/\{+1\}$ by $g(z):=(az+b)/(cz+d)$ for $z\in\mathbb{H}$ and $g=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ mod(±1).

ii) Since $\pi_1(PSL(2,R)) = \mathbb{Z}$, the universal covering map defines a cyclic extension.

(5.2.1) $$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{PSL}(2,\mathbb{R}) \longrightarrow PSL(2,\mathbb{R}) \longrightarrow 1 \qquad \text{(exact)}.$$

An element $\tilde{g}$ of $\widetilde{PSL}(2,\mathbb{R})$ is represented by a pair $(g,\varphi(z))$ of an elment g of PSL(2,$\mathbb{R}$) and a branch $\varphi(z)$ of the function $\log((cz+d)^2)/2\pi\sqrt{-1}$ on $\mathbb{H}$. The product is given by $\tilde{g}\circ\tilde{h} = (g\circ h,\psi(z)+\varphi(h(z)))$ for $\tilde{g} = (g,\varphi(z))$ and $\tilde{h} = (h,\psi(z))$.

$\widetilde{PSL}(2,\mathbb{R})$ acts on the infinite cyclic covering $\mathbb{H}_\infty$ of the cannonical $\mathbb{C}^*$-bundle of $\mathbb{H}$.

(5.2.2) $$\tilde{g}(z,\lambda) = (g(z),\ \lambda+\varphi(z)) \quad \text{for} \quad (z,\lambda)\in\mathbb{H}_\infty\cong\mathbb{H}\times\mathbb{C}$$
$$\text{and} \quad \tilde{g} = (g,\varphi(z)) \in \widetilde{PSL}(2,\mathbb{R}).$$

iii) For a positive integer d, (5.2.1) induces a finite cyclic extention,

(5.2.3) $$1 \longrightarrow \mathbb{Z}/\mathbb{Z}d \longrightarrow \widetilde{PSL}(2,\mathbb{R})/\mathbb{Z}d \longrightarrow PSL(2,\mathbb{R}) \longrightarrow 1 \quad \text{(exact)}.$$

An element $\tilde{g}$ of $\widetilde{PSL}(2,\mathbb{R})/\mathbb{Z}d$ is represented by a pair $(g,\varphi(z))$ of an element g of PSL(2,$\mathbb{R}$) and a branch $\varphi(z)$ of the function $(cz+d)^{2/d}$ on $\mathbb{H}$. The product is $g\circ h = (g\circ h,\psi(z)\varphi(h(z)))$ for $\tilde{g} = (g,\varphi(z))$ and $\tilde{h} = (h,\psi(z))$.

The group $\widetilde{PSL}(2,\mathbb{R})/\mathbb{Z}d$ acts on the C*-bundle $\mathbb{H}_d := \mathbb{H}_\infty/\mathbb{Z}d$ over $\mathbb{H}$.

(5.2.4) $\tilde{g}(z,v) = (g(z), v\varphi(z))$ for $(z,v) \in \mathbb{H}_d \cong \mathbb{H}\times\mathbb{C}^*$, and $\tilde{g} = (g, \varphi(z)) \in \widetilde{PSL}(2,\mathbb{R})/\mathbb{Z}d$.

The action of $PSL(2,\mathbb{R})/\mathbb{Z}d$ on $\mathbb{H}$ does not have a fixed point. (If (z_0, v) were a fixed point of $(g, \varphi(z))$, then z_0 is an elliptic fixed point of g such that $\varphi(z_0)=1$.)

Note. Recalling the fact $dg(z)/dz = (cz+d)^{-2}$, it is easy to see that the d-th power of the C*-bundle $\mathbb{H}_d$ over $\mathbb{H}$ is the cannonical C*-bundle of $\mathbb{H}$.

(5.3) A splitting factor for a finite cyclic extension of a Fuchsian group.

Let $\Gamma \subset PSL(2,\mathbb{R})$ be a co-compact Fuchsian group of the first kind. Let Γ_d be the inverse image of Γ in $PSL(2,\mathbb{R})/\mathbb{Z}d$ by the map (5.2.3) so that

(5.3.1) $$1 \longrightarrow \mathbb{Z}/\mathbb{Z}d \longrightarrow \Gamma_d \longrightarrow \Gamma \longrightarrow 1 \quad \text{(exact)}.$$

A splitting factor of the sequence (5.3.1) is a subgroup Γ^* of $PSL(2,\mathbb{R})/\mathbb{Z}d$ which is bijective to its image Γ. The projection map from $\tilde{g}=(g,\varphi(z)) \in \Gamma^* \cong \Gamma$ to its second factor $\varphi(z)$ define's an automorphic factor, discussed in [8],[35,(3.1.2)].

Note 1. The sequence (5.3.1) does not split in general. Even it does split, the splitting is not unique, but depends on d-torsions of the Picard variety of $\mathbb{H}/\Gamma$.

Note 2. If d=2, the sequence (5.2.3) and the C*-bundle $\mathbb{H}_2$ are rewritten as,

(5.3.2) $$1 \longrightarrow (\pm I) \longrightarrow SL(2,\mathbb{R}) \longrightarrow PSL(2,\mathbb{R}) \longrightarrow 1 \quad \text{(exact)},$$

(5.3.3) $$\mathbb{H}_2 \cong \mathbb{H}\times\mathbb{C}^* \simeq \tilde{\mathbb{H}} := ((u,v) \in \mathbb{C}^2 : \operatorname{Im}(u/v) > 0)$$
$$(z,v) \longmapsto (zv, v)$$

so that the linear action of $SL(2,\mathbb{R})$ on $\tilde{\mathbb{H}}$ induces the action (5.2.4). Hence the splitting facter is nothing but a co-compact subgroup Γ of $SL(2,R)$ such that $\Gamma \not\ni \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$.

(5.4) The Gorenstein singular point with good C*-action ([8],[21],[34]).

Let $\Gamma^* \subset PSL(2,\mathbb{R})/\mathbb{Z}d$ be a splitting factor of (5.3.1), which acts on $\mathbb{H}$ proper and fixed point free so that $\mathbb{H}_d/\Gamma^*$ is a complex two manifold. By adding a point, put

(5.4.1) $$X_0 := (0) \cup \mathbb{H}_d/\Gamma^*.$$

1. X_0 <u>has naturally a structure of affine algebraic variety with an isolated normal singular point at 0 such that</u>

i) X_0 admitts a good C*-action (ie $0 \in X_0$ is in the closure of every orbit [19].)

ii) X_0 is normal Gorenstein variety so that there is no-where vanishing holomorphic 2-form ω on $X_0-(0)$ such that the C*-action induces,

(5.4.2) $\qquad t^*(\omega) = t^{-d}\omega$, $\quad$ for $t \in \mathbb{C}-(0)$,

2. Conversely if X_0 is a two dimentional variety with an isolated singular point 0 satisfying the above i), ii) and $d > 0$, then it is expressed as (5.4.1) for a suitable Fuchsian group Γ and its splitting factor Γ^*.

Proof. i) Let Γ' be a finite index normal subgroup of Γ, which has no fixed point on $\mathbb{H}$ (cf [3],[10]) and let Γ'^* be the corresponding subgroup of Γ^*. Then $\mathbb{H}_d/\Gamma'^*$ is a C*-bundle over $\mathbb{H}/\Gamma'$ whose associated line bundle $(\mathbb{H}/\Gamma')\cup(\mathbb{H}_d/\Gamma'^*)$ is negative, since its d-th power is the cannonical bundle of the curve $\mathbb{H}/\Gamma'$ (cf (5.2) Note.). Hence the zero-section $\mathbb{H}/\Gamma'$ of the bundle can be blow down to a point 0, to obtain an affine variety $(0)\cup\mathbb{H}_d/\Gamma'^*$, on which still the finite group $\Gamma/\Gamma'=\Gamma^*/\Gamma'^*$ acts in a natural manner where 0 is the only fixed point of the action. Thus $(0)\cup\mathbb{H}_d/\Gamma^*$ $= ((0)\cup\mathbb{H}_d/\Gamma'^*)/(\Gamma/\Gamma')$ naturally obtains a structure of an affine variety with an isolated singular point at 0, which is normal by definition.

ii) The C*-action on the bundle $\mathbb{H}_d/\Gamma^*$ naturally induces the good C*-action on X_0.

iii) The holomorphic two form on $\mathbb{H}_d$ of the following form:

(5.4.3) $\qquad \omega := dzdv/v^d$

is invariant by the action of $\widetilde{PSL}(2,\mathbb{R})/\mathbb{Z}d$ (5.2.4). Hence it induces a nowhere vanishing holomorphic two form on $X_0-(0) = \mathbb{H}_d/\Gamma^*$, denoted again by ω. Since the singularity X_0 is normal two dimensional, it is Macaulay. These imply that X_0 is Gorenstein. The (5.4.2) follows, since the form (5.4.3) satisfies the same formula. The fact that exponent $-d$ in (5.4.3) is $=< 1$ implies that X_0 cannot be smooth.

2. Due to Pinkham [21] (compaire also [4],[11]), there exists a finite covering X_0' of X_0 ramifying only at 0, s.t. X_0' is obtained by blowing down of the zero section section of a negative line bundle over a curve C. X' is still Gorenstein and the existence of a non-vanishing holomorphic two form implies that a power of the line bundle is the cannonical bundle of the curve C([8,Prop.1], [23,(5.)]). That $d > 0$ implies that Euler number of $C < 0$. Uniformizing the curve C by $\mathbb{H}$ gives the proof.
q.e.d.

(5.5) Hypersurface case.

1. i) The germ of X_0(5.4.1) near at 0 can be analytically embedded in $\mathbb{C}^3$, iff X_0 is globally embeded in $\mathbb{C}^3$ as a hypersurface for a weighted homogeneous polynomial f.

(5.5.1) $X_c := \{ (x,y,z) \in \mathbb{C}^3 : f(x,y,z) = 0 \}$,

(5.5.2) $f(x,y,z) = \sum_{ai+bj+ck=h} c_{ijk} x^i y^j z^k$.

Here weights a,b,c and h are positive integers such that

(5.5.3) $0 < a,b,c \leq h/2$, $GCD(a,b,c,h) = 1$ and $d = h-a-b-c$.

ii) Up to a constant factor, the form ω (5.4.3) is identified with the form,

(5.5.4) $\omega := Res[dxdydz/f(x,y,z)]$

2. For given weights (a,b,c;h), there exist at least one polynomial (5.5.2) having an isolated critical point at 0, iff the following rational function $\chi(T)$, may have poles only at T=0. Its Laurent expantion at T=0 has non-negative coefficients [23].

(5.5.5) $$\chi(T) := T^{-h} \frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)}$$

Proof. 1. Suppose the germ $(X_c, 0)$ is given by the hypersurface g=0 for a $g \in \mathbb{C}\{x,y,z\}$. The existence of a C*-action on X_0 implies that g belongs to the ideal $(\partial g/\partial x, \partial g/\partial y, \partial g/\partial z)$ in $\mathbb{C}\{x,y,z\}$. Then there exists a local coordinate change, which brings g to a polynomial of the form (5.5.2) ([25]). The local isomorphism of the surface X_0 (5.4.1) and the hypersurface (5.5.1) extends to a global isomorphism since both surfaces admit unique good C* actions. Since X_0 is normal, the proportion $Res[dxdydz/f(x,y,z)]/\omega$, which is holomorphic nowhere vanishing on $X_0 - \{0\}$, extends to a unit function on X_0. Hence a+b+c+d-h = 0.

Note. For a fixed (a,b,c;h), the set of polynomials having isolated critical point at 0 is Zariski open in the set of all polynomials of the form (5.5.2).

Definition [23] 1. A system of positive integers (a,b,c;h) with $\max(a,b,c) \leq h$ is called regular if the function $\chi(T)$ (5.5.5) may have poles at most at T = 0. It is called reduced if gcd(a,b,c,h) = 1 except for the type A (cf [24,(.5)]).

2. Let us develop $\chi(T)$ in the finite Laurent series of the form,

(5.5.6) $\chi(T) = T^{m_1} + T^{m_2} + \dots + T^{m_\mu} = \sum_m a_m T^m$.

We call $m_1, \dots, m_\mu$ the exponents for $(a,b,c;h)$ and a_m the multiplicity of the exponent m . We have $\mu = \sum_m a_m$. The smallest exponent $(= a+b+c-h)$ is denoted by ε. In case $\varepsilon < 0$, we shall also use a notation $d := -\varepsilon = h-a-b-c$ (cf (5.5) 1. i)).

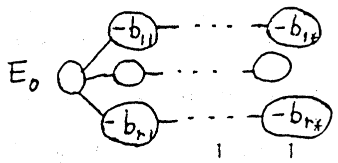
Let $(a,b,c;h)$ be any reduced regular system of weights. Then there exists always an exponent either equal to 1 or -1 [24]. Hence if $\varepsilon \neq \pm 1$, we have an innequality

(5.5.7) $d + 1 \geqq \min(a,b,c)$.

(5.6) Resolutions of the singlarity.

The minimal good resolution $\pi: \tilde{X}_0 \to X_0$ of X_0 at 0 is described as follows[6],[19],[21]

i) Let Γ and Γ^* be the Fuchsian group and the splitting factor for X_0 (5.3). There is a natural map from the quotient variety $\hat{X}_0 := (\mathbb{H} \cup \mathbb{H}_d)/\Gamma^* = \mathbb{H}/\Gamma \cup \mathbb{H}_d/\Gamma^*$ to $X_0 = (0) \cup \mathbb{H}_d/\Gamma^*$, which is the weighted blowing up of X_0 at 0 and $\mathbb{H}/\Gamma$ is its exceptional set.Then $\hat{X}_0$ has a cyclic quotient singularity of type (p,d_*) at $x \in \mathbb{H}/\Gamma \subset \hat{X}_0$, where x is a fixed point of Γ by an isotropy subgroup of order p and d is an integer s.t. $d_* = d \bmod(p)$ and $0<d_*<p$. By resolving such cyclic quotient singularities on $\hat{X}_0$ minimally, we obtain the minimal good resolution $\tilde{X}_0$ of X_0. The strict transform of $\mathbb{H}/\Gamma$ in $\tilde{X}_0$ is denoted by E_0 and called the central curve. Let $A := (p_1, \dots, p_r)$ be the set of the orders of isotropy subgroups. Then the dual graph of the resolution (defined in [19]) is as follows. Obviously the graph is branching at the fixed points on $\mathbb{H}/\Gamma \cong E_0$.

(5.6.1) 

} r- branches ,

(5.6.2) $p_i/d_{*i} = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{b_{i3} \cdots - \cfrac{1}{b_{i*}}}}$ (continued fraction), $i = 1, \dots, r$.

In case X_0 is a hypersurface for the weights $(a,b,c;h)$, E_0 is identified with the curve in $\mathbb{P}(a,b,c)$ defined by the equation $f = 0$ and the branching points set is a subset of the intersection of E_0 with the coordinate axis of $\mathbb{P}(a,b,c)$. Then,

(5.6.3) $g(E_0) = a_0$, (Here $g(E_0)$ means the genus of E_0.)

(5.6.4) $-E_0 \cdot E_0 = a_1 - a_0 + 1$. (Here $E_0 \cdot E_0$ means the self-intersection number of E_0.)

(5.6.5) $A = (e \in (a,b,c) : e \nmid h) \cup (\gcd(e,f) * (N(e,f)-1) : (e,f) \in (a,b,c)\text{-diagonal subset})$

Here

$$N(e,f) := \frac{1}{h!}\left(\frac{\partial}{\partial T}\right)^h \frac{1}{(1-T^e)(1-T^f)}\Bigg|_{T=0}$$ and $s*t :=$ t-copies of s.

(Exactly the set A (5.6.5) presents the set of orders of isotropy groups at the point of $E_0\cap$(coordinate axis of $\mathbb{P}(a,b,c)$). Hence 1 must be deleted from A if it appears.) The $\mathrm{Vol}(\Gamma)/2\pi := 2(g(E_0)-1) + \sum_{i=1}^{r}(1-1/p_i)$ of the fundamental domain for Γ is given by

(5.6.6) $$\mathrm{Vol}(\Gamma)/2\pi := d\,\frac{h}{abc}$$.

(The formula is shown similarly to the case $d = \pm 1$ [23].)

ii) Let the cannonical divisor K_0 on $\tilde{X}_0$ of the singularity $0\in X_0$ be defined as,

(5.6.7) $K_0 := \mathrm{div}(\pi^*(\omega)) :=$ the zeros minus poles of the lifted 2-form $\pi^*(\omega)$ on $\tilde{X}_0$.

In fact $\pi^*(\omega)$ does not have zeros for a minimal good resolution so that $-K_0$ is effective (Tomari, unpublished). The coefficients of E_0 in K_0 is equal to $\varepsilon - 1$.

(5.7) The universal unfolding for $f(x,y,z)$ and the Milnor fiber.

i) The universal unfolding of $f(x,y,z)$ (Thom [ ]) is defined as a polynomial

(5.7.1) $$F(x,y,z,t_1,t_2,\ldots,t_\mu)$$

such that $f(x,y,z) = F(x,y,z,0,\ldots,0)$ and the partial derivatives $\dfrac{\partial F(x,y,z,0,\ldots,0)}{\partial t_i}$ ($i=1,\ldots,\mu$) form a C-bases of the Jacobi ring $\mathbb{C}[x,y,z]/(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$. Since the Jacobi ring is graded ring, whose Poincare polynomial is equal to $T^{-\varepsilon}\chi(T)$, we may assume that F is a weighted homogeneous polynomial of degree h with respect to $\deg(x)=a$, $\deg(y)=b$, $\deg(z)=c$ and $\deg(t_i)=m_i+\varepsilon$ ($i= 1,\ldots,\mu$).

Denote by m_-, m_0 and m_+ the number of parameters t_i, whose degree is negative, zero, and positive respectively. By definition,

(5.7.2) $$m_- = \sum_{m<-\varepsilon} a_m, \quad m_0 = a_{-\varepsilon}, \quad m_+ = \sum_{m>-\varepsilon} a_m \quad \text{and} \quad \mu = m_- + m_0 + m_+ .$$

The equation $F = 0$ defines a family of affine algebraic surfaces

(5.7.3) $$X_t := \{ (x,y,z)\in\mathbb{C}^3 : F(x,y,x,t) = 0 \} \qquad \text{for } t:=(t_1,\ldots,t_\mu)\in\mathbb{C}^\mu.$$

Particularly $(X_t,0)$ for $t\in\mathbb{C}^{m_-}\times\mathbb{C}^{m_0}\times 0$ defines a family of equisingularities. The family X_t for $t\in 0\times\mathbb{C}^{m_0}\times\mathbb{C}^{m_+}$ is studied by many authers since the surfaces are naturally completed by adding a divisor at infinity as we see in (5.8).

ii) Let us denote by S (resp. S_f) the Zariski open subset of $0\times\mathbb{C}^{m_0}\times\mathbb{C}^{m_+}$ consisting of points t s.t. X_t has at most finite number of (resp. rational) singularities.

A smooth fiber X_t over $t\in S$ is called a Milnor fiber, whose middle homology $H_2(X_t, \mathbb{Z})$ is a free abelian group of rank μ with the intersection form I of sign (μ_+, μ_c, μ_-).

(5.7.4) $\qquad \mu_+ = 2\sum_{m<0} a_m = 2\sum_{m>h} a_m, \qquad \mu_c = 2a_0 = 2a_h, \qquad \mu_- = \sum_{0<m<h} a_m.$

iii) The geometic genus $p_g(X_t, 0)$ of X_t at 0 for $t\in \mathbb{C}^{m_-}\times\mathbb{C}^{m_0}\times 0$ is defined as $H^1(\tilde{X}_t, \mathcal{O}_{\tilde{X}_t})$ for a resolution $\tilde{X}_t \longrightarrow X_t$ of the singular point 0. Then, we have a formula ([27],[9]),

(5.7.5) $\qquad p_g(X_t, 0) = (\mu_+ + \mu_c)/2 = \sum_{m\le 0} a_m .$

iv) a) X_t is rational. $\underset{def.}{\iff}$ $p_g(X_t, 0) = 0$ $\iff$ All exponents are positive. $\iff$ $\varepsilon = 1$.
b) X_t is minimally elliptic. $\underset{def.}{\iff}$ $p_g(X_t, 0) = 1$ $\iff$ ε is the only non-positive exponent.

(5.8) The family of compact surfaces over S .

i) Define the weighted homogneous polynomial $G(x,y,z,w)$ of weights $(a,b,c,1)$, and the compact hypersurface $\overline{X}_t$ in $\mathbb{P}(a,b,c,1)$ with parameter $t\in S$.

(5.8.1) $\qquad G(x,y,z,w,t) := w^h\, F(x/w^a, y/w^b, z/w^c, 0, \ldots, 0, t_{m\pm 1}, \ldots, t_\mu)$,

(5.8.2) $\qquad \overline{X}_t := \{(x:y:z:w)\in \mathbb{P}(a,b,c,1) : G(x,y,z,w,t) = 0\}$ for $t\in S$.

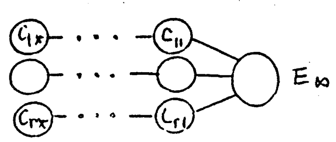
$\overline{X}_t$ is a C* equivariant comactification of X_t such that the complement $E' := \overline{X}_t - X_t$ is a curve isomorphic to E_0. The surface $\overline{X}_t$ has cyclic quotient singularities of type $(p, p-d_*)$ for $p\in A$ along E'. The family (5.8.2) is analytically trivial near E' so that the singularities can be resolved simultaneously for $t\in S$.

ii) Denote by $\tilde{\overline{X}}_t$ the smooth surface obtained by resolving the singular points of $\overline{X}_t$ minimally. Let us decompose $\tilde{\overline{X}}_t$ as,

(5.8.3) $\qquad \tilde{\overline{X}}_t = \tilde{X}_t \cup D_\infty .$

Here $\tilde{X}_t$ is the minimal resolution of the affine variety X_t and $D_\infty := \tilde{\overline{X}}_t - \tilde{X}_t$, called the divisor at infinity. The strict transform of E' in $\tilde{\overline{X}}_t$ will be denoted by E_∞ and called the central curve of D_∞.

The dual graph of the diviser D is as follows,

(5.8.4) r-branches 

(5.8.5) $\quad p_i/(p_i - d*_i) = c_{i1} - \cfrac{1}{c_{i2} -} \cfrac{1}{c_{i2} \cdots -} \cfrac{1}{c_{i*}}$ (continued fraction). $(i=1,\cdots,r)$

(5.8.6) $\quad -E_\infty^2 = r - a_1 + a_0 - 1$.

iii) The cannonical diviser $K_{\tilde{\tilde{X}}_t}$ of $\tilde{\tilde{X}}_t$ is calculated as follows.

(5.8.7) $\quad K_{\tilde{\tilde{X}}_t} = K_\infty + \sum_{x \in X_t} K_x$,

where a) K_x is the cannonical divisor of the singularity x of the affine surface X_t.

b) K_∞ is the diviser having the support on D_∞, whose coefficients of E_∞ is d-1 satisfying the adjunction relation: $2g(E)-2 = K_\infty E + E$ for the curves E on D_∞.

Particularly for $\underbrace{t \in S_f}_{(5.7)\,ii)}$, the second term vanishes so that we obtain,

(5.8.8) $\quad K_{\tilde{\tilde{X}}_t} = K_\infty \quad$ for $\quad t \in S_f$.

(Proof of iii). A cannonical divisor $K_{\tilde{\tilde{X}}_t}$ of $\hat{X}_t$ is given by the zeros and poles of a two from on $\bar{X}_t$ induced from $\mathrm{Res}_{\bar{X}_t}\left[\dfrac{(axdydz + bydzdx + czdxdy)dw + wdxdydz}{w^{1+\varepsilon} G(x,y,z,w,t)}\right]$

, which is regular and non-zero on X_t and is zero of order d-1 along E'_∞.)

(5.9) Middle homology groups of X_t and $\tilde{\tilde{X}}_t$.

Let $\tilde{\tilde{X}}_t$ be any smooth surface obtained by blowing down some exceptional curves contained in D_∞ and let us denote by $\tilde{\tilde{D}}$ the blow down image of D_∞ in $\tilde{\tilde{X}}_t$.

1. The surface $\tilde{\tilde{X}}_t$ for $t \in S_f$ is simply connected. Hence the first Betti number b_1 and the irregularity $q := \dim H^1(\tilde{\tilde{X}}_t, \mathcal{O}_{\tilde{\tilde{X}}_t})$ of the surface are zero.

2. The natural inclution $\tilde{X}_t \subset \tilde{\tilde{X}}_t$ induces an isomorphism of lattices,

(5.9.1) $\quad H_2(\tilde{X}_t, \mathbb{Z})/\mathrm{rad}(I) = (\mathbb{Z}[\tilde{\tilde{D}}])^\perp, \quad$ for $t \in S_f$.

Here $\quad \mathrm{rad}(I) := \{ e \in H_2(\tilde{X}_t, \mathbb{Z}) : I(e,x) = 0 \text{ for } x \in H_2(\tilde{X}_t, \mathbb{Z})\}$,

$\mathbb{Z}[\tilde{\tilde{D}}]$:= the submodule of $H_2(\tilde{\tilde{X}}_t, \mathbb{Z})$ generated by the homology classes $[E_i]$ for irreducible components E_i of $\tilde{\tilde{D}}$.

3. Homology classes for irreducible components of $\tilde{\tilde{D}}$ are linearly independent.

(5.9.2) $\quad \mathrm{disc}\, \mathbb{Z}[\tilde{\tilde{D}}] = \pm\, \mathrm{disc}\, H_2(\tilde{X}_t, \mathbb{Z})/\mathrm{rad}(I)$,

(5.9.3) $\quad \mathrm{rank}\, H_2(\tilde{\tilde{X}}_t, \mathbb{Z}) = \mu - \mu_0 + \#\{\text{irreducible components of } \tilde{\tilde{D}}\}$.

×7

Proof. 1. Due to a theorem of Brieskorn [2], the resolution $\widetilde{X}_t$ of rational double point is homeomorphic to a smooth fiber, say $X_{t'}$. Hence we have only to prove for the case when X_t is a smooth Milnor fiber. Since $\widetilde{\widetilde{D}} = \widetilde{\widetilde{X}}_t - X_t$ has real codimension 2 in $\widetilde{\widetilde{X}}_t$, one has an epimorphism $\pi_1(X_t) \longrightarrow \pi_1(\widetilde{\widetilde{X}}_t)$. The Milnor fiber X_t is simply connected.

2.,3. We have only to consider the case $\widetilde{\widetilde{X}}_t = \widetilde{X}_t$ due to the follwing:

Let S be a smooth surface with an exceptional curve E of the first kind. Put $\widetilde{S} = S/E$. Then we have isomorphisms $H_2(S,\mathbb{Z}) = (\mathbb{Z}[E])^{\perp}$ of lattices.

The natural inclution map $X_t \subset \widetilde{\widetilde{X}}_t = X_t \cup D_\infty$ induces a homomorphism,

$$(5.9.4) \qquad H_2(X_t,\mathbb{Z}) \longrightarrow H_2(\widetilde{X}_t,\mathbb{Z})$$

, which is a part of the following long exact sequence,

$$0=H_3(\widetilde{X}_t,\mathbb{Z}) \longrightarrow H_3(\widetilde{X}_t,X_t,\mathbb{Z}) \longrightarrow H_2(X_t,\mathbb{Z}) \longrightarrow H_2(\widetilde{X}_t,\mathbb{Z}) \longrightarrow H_2(\widetilde{X}_t,X_t,\mathbb{Z}) \longrightarrow H_1(X_t,\mathbb{Z}) = 0.$$

Here $H_3(\widetilde{X}_t,X_t,\mathbb{Z}) \cong H^1(D_\infty,\mathbb{Z}) \cong H^1(E_\infty,\mathbb{Z})$ and $H_2(\widetilde{X}_t,X_t,\mathbb{Z}) \cong H^2(D_\infty,Z) = \mathbb{Z}[D_\infty]$.

The map $H\,(E,\mathbb{Z}) = H\,(E\,,\mathbb{Z}) \longrightarrow H_2(X_t,\mathbb{Z})$ is obtained by associating to a cycle $c \in H\,(E\,,Z)$ the total space of a S^1-bundle $l(c)$ over c (= the boundary of the normal disc-bundle of c in $\widetilde{X}_t$).

The map $H_2(\widetilde{X}_t,\mathbb{Z}) \longrightarrow H^2(D_\infty,\mathbb{Z}) = \mathbb{Z}[D_\infty]$ is obtained by taking the cap products with the homology classes $[E_i]$ of the irreducible components E_i of D_∞. Hence the kernel of the map is $(\mathbb{Z}[D_\infty])^{\perp}$. The surjectivity of the map implies the linear independence of irreducible components of D_∞ and hence $\mathrm{rank}(\mathbb{Z}[D_\infty])^{\perp}$ $= \mathrm{rank}\, H_2(\widetilde{X}_t,\mathbb{Z}) - \#$ irreducible components of D_∞.

Since the map (5.9.4) is metric preserving so that its kernel $H_1(E_\infty,\mathbb{Z})$ is contained in rad(I). Thus we obtain a surjection, $(\mathbb{Z}[D_\infty])^{\perp} \longrightarrow\!\!\!\!\!\rightarrow H_2(X_t,\mathbb{Z})/\mathrm{rad}(I)$.

The Euler number $c_2(\widetilde{X}_t)$ of the compact surface $\widetilde{X}_t$ is calculated as

$$c_2(\widetilde{X}_t) = \text{Euler number of } X_t + \text{Euler number of } D_\infty$$
$$= (1+\mu) + (2 - 2g(E_\infty)) + \#\{\text{irreducible components of } D_\infty - E_\infty\}$$

Recalling $c_2(\widetilde{X}_t) = 2 +$ the second Betti number of $\widetilde{X}_t$ and $g(E_\infty) = g(E_0)$, we get an equality $\mathrm{rank}H_2(X_t,Z)/\mathrm{rad}(I) = \mathrm{rank}(Z[D_\infty])^{\perp}$, which implies the isomorphism (5.9.1).qed

Note. The above calculation shows also the bijection of the modules,

$$(5.9.5) \qquad \mathrm{rad}(I) = H_1(E_\infty,Z)\ .$$

References

[1] Arnold, V.I.:

[2] Brieskorn

[3] Brieskorn, E. : Singular elements of semi-simple algebraic groups. Proc. Interna

[4] Bungaard, S., Nielsen, J. : On normal subgroups with finite index in F-groups. M

[5] Demazure, M. : Surfaces de Del Pezzo II,III,IV,V. Seminaire sur les singularites

[6] Dolgachev, I.: Quotient-conical singularitieson complex surfaces. Funct. Anal. A

[7] Dolgachev, I.: Automorphic forms ans quasihomogeneous singularities. Funct. Anal

[8] Dolgachev, I.: On the link space of a Gorenstein quasihomogeneoussurface singula

[9] Durfee, A. H. : The Signature of Smoothing of Comples Surface Singularities.

[10] Ebeling, W. and Wall, C.T.C. : Kodaira singularities and an extension of Arnold'

[11] Fox, R.H. : On Fenchesl's conjecture about F_groups, Matematisk Tidsskrift B, 61

[12] Hirzebruch, F. :

[13] Horikawa, E. :

[14] Laufer, H. : On Minimally Elliptic Singularities. Amer.J.Math.,99 (1977),1257-12

[15] Looijenga, E. : Smoothing components of a triangle singularity.I, Proc.of Sympos

[16] Looijenga, E. : Smoothing components of a trangle singularity.II,

[17] Milnor,J. and Orlik,P. : Isolated singularities defined by weighted homogeneous

[18] Oda, T. : K. Saito's period map for holomorphic functions with isolated critical

[19] Orlik, P.,Wagreich, P.: Isolated singularities of algebraic surfaces with C*-act

[20] Pinkham, H.: Singularites exceptionelles, la dualite etrange d'Arnol'd et les su

[21] Pinkham, H.: Normal surface singularities with C*-action.Math. Ann. 227, 183-193

[22] Pinkham, H.:Deformations of normal surface singularities with C*-action. Math. A

[23] Saito,K. : Regular System of weights and associated singularity, to appear in Ad

[24] Saito,K. : On the existence of exponents prime to the Coxeter number. RIMS prep

[25] Saito,K. : Quasihomogene isolierte Singularitaten von HyperfLachen.Inventiones m

[26] Saito,K.: Extended affine root systems I (Coxeter transformations). Publ. R.I.M.

[27] Saito,Mo. : On the exponents and the geometric genus of an isolated hypersurface

[28] Saito,Ma., Shimizu,Y. and Usui,S. : Vanishing of mixed Hodge structure and Torel

[29] Shiota,T.:

[30] Steenbrink, J. H. M. : Mixed Hodge structure on the vanishing cohomology. Real a

[31] Todorov, A. : Mat. Zametki USSR (1974)

[32] Todorov, A. : A construction of surfaces with p = 1, q = 0 and 2 < K < 8, Counte

[33] Wahl, J. : Elliptic deformations of minimally elliptic singularities, Math. Ann.

[34] Wahl, J. : A characterization of Quasi-homogeneous Gorenstein surface singularit

[35] Wagreich, P.: Algebra of automorphic forms with few generaters. Trans. AMS, 626

[36] Wagreich, P.: The structure of quasihomogeneous singularities. Proc. Symp. Pure

[37] Watanabe, K-I. :

g(E)

E0 E0

Ef Ef

C = 22 - Milnor

d (1 + # infinity curves) = | A | + r - 2